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Research progress on applications of polynomial optimization

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1 Introduction

The goal of this work package is to develop new algebraic techniques for polynomial optimization. To reach this goal, we aim to take advantage of the interplay with algebra, semidefinite programming, moment and/or sums of squares decomposition techniques. This new approach should allow us to obtain either more reliable numerical solvers for polynomial optimization and/or efficient algorithms based on symbolic computation which are our primary goals. These algorithms should also be specialized to exploit particular structures (Work Package 3) and deal with applications that were not tractable previously (Work Package 4). It is divided in 4 tasks.

Task 1.1 Algebraic tools and Semidefinite Programming (led by SU). As sketched above, semidefinite programming is central in polynomial optimization and symbolic computation is crucial for reliability. In this task, we will aim to develop algebraic techniques that help in using SDP solvers for POPs in a better way, or improving SDP solvers or developing exact algorithms for solving POP problems. A first approach consists in combining algebraic techniques and moment relaxations to reduce the size of SDP problems when translating original POPs to SDPs. A second approach consists in studying the algebraic properties of geometric objects of POP. From this one could expect new exact complete algorithms for POP that could be used when all classical assumptions of SDP based algorithms are not satisfied (or for ill-conditioned problems).

Task 1.2 Certificates of positivity (led by UKON). SDPs are mainly used to compute certificates of positivity by means of sums of squares (SOS). Most of the current practical approaches are based on numerical solving of the SDP, which provides approximate certificates of positivity. We will aim to develop algebraic techniques to reduce multivariate problems to univariate ones for which one can develop fast implementations of exact algorithms. We will also aim to develop algebraic ideas of how to set up an SDP relaxation for a tight polynomial inequality (i.e. an inequality holding with equality on some points) so that a numerically obtained sums of squares certificate for this inequality to hold can be turned into a rational certificate. Up to now there was no systematic way to get rational certificates for tight inequalities.

Task 1.3 Tensor decompositions (led by UNIFI). Tensor decomposition is an active area of research, with applications in signal processing, phylogenetics, machine learning and other areas. Exact tensor decomposition algorithms have already been introduced by members of the project; we already emphasized the central role of it for the truncated moment problem. These algorithms are based on algebraic geometry techniques, namely on contraction maps defined by suitable vector bundles. The next step is to convert these algorithms into numerical approximate algorithms, in order to use them in situations of noisy data, in case it comes from statistical sampling or is just known approximation approximately. To do that, we need to find kernels of structured matrices of low rank, which are close to the matrix coming from initial data. This sub-problem still requires algebraic studies and a theory development.

Task 1.4 Non-commutative polynomial optimization (led by NWO-I). Polynomial optimization in non-commuting variables offers the right setting for attacking hard problems in control (eigenvalue optimization) and quantum information (with positive operators and projections being the natural variables in quantum measurements). We propose to investigate hierarchies of semidefinite approximations based on non-commutative polynomial algebra. Understanding their convergence properties is closely related to a long standing open conjecture of Connes in operator algebra. We will also aim to study polynomial system solving over non-commuting real variables. In the recent years we have contributed, together with other partners of this network, to some numerical methods based on combining the sums of squares and moment methodologies for polynomial optimization. A research direction is to investigate whether these methods can be extended to solve systems of equations in non-commuting variables, where the objective would be to find solutions that are positive operators. Such questions arise indeed naturally within the quantum setting, with quantum graph isomorphism as a typical example.

2 Status of research performed by ESRs

2.1 ESR1 - Vu Trung Hieu/SU

Paper 1: Victor Magron, Mohab Safey El Din, Trung-Hieu Vu; *Sum of Squares Decompositions of Polynomials over their Gradient Ideals with Rational Coefficients*

Consider a polynomial f in n variables with rational coefficients, $f \in \mathbb{Q}[x_1, \ldots, x_n]$. Decomposing a multivariate polynomial with rational coefficients as a sum of squares (SOS) of polynomials provides a certificate for its nonnegativity. This certification task for the multivariate case is especially hard because there exist nonnegative polynomials that are not SOS, and even if they are, there is no guarantee that there exists an SOS decomposition involving rational coefficients. We recall that, for the univariate case, $f \in \mathbb{Q}[x_1]$ is nonnegative if and only if f is SOS in $\mathbb{Q}[x_1]$. Further, there already exist two algorithms to handle the univariate case.

We introduce a key to solve this problem for the multivariate case. Under the conditions that the infimum of f over $\mathbb{R}$ is attained and the gradient ideal of f is zero-dimensional radical, we prove that f is nonnegative over $\mathbb{R}^n$ if and only if f is SOS modulo the gradient ideal of f over $\mathbb{Q}[x_1, \ldots, x_n]$, i.e. f can be decomposed as follows $f = \sum_{j=1}^{s} q_j^2 + \sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i}$, for some polynomials $q_1, \ldots, q_s, \phi_1, \ldots, \phi_s \in \mathbb{Q}[x_1, \ldots, x_n]$, where $\frac{\partial f}{\partial x_i}(i = 1, ..., n)$ are partial derivatives of f.

The proof is based on the following idea: reducing the dimension of the space we are considering. Algebraically, this allow us to reduce the number of variables. Then, the result on rational SOS decomposition of univariate polynomials can be applied.

Based on the proof, we propose an algorithm to find the rational SOS decomposition for f that has the ability to certify non-negativity on problems which cannot be tackled with a direct SOS approach. To build the algorithm, we need subroutines concerning to finding a shape position of an ideal having zero-dimentional, finding quotients of the division of a polynomial by a list of polynomials in shape position, and finding an SOS decomposition of a non-negative univariate polynomial. We investigate also the complexity analysis of the algorithm. Assume that the input are a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ in n-variate, of degree d and bitsize τ, and a given lexicographic monomial order. We prove that the algorithm runs in $O^{\sim}((\tau+n+d)^2 d^{6n} + (d^{n+1}/2)^{3d^{n+1}/2}(\tau+n+d)d^{3n+1})$ or $O^{\sim}((\tau+n+d)^2 d^{6n} + n^3(\tau+n+d)d^{9n+3})$ boolean operations, it depends on using Algorithm `univsos1` or Algorithm `univsos2`, respectively, in [MSS19] to compute an SOS decomposition of a nonnegative univariate polynomial in the algorithm. The paper is in preparation and will be submitted during December.

Paper 2: Trung-Hieu Vu; *On the solution existence and stability of polynomial optimization problems*

We consider the minimization problem $\mathrm{OP}(K, f)$, where K is a nonempty closed subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial in n variables of degree $d \geq 2$. Let f_d be the homogeneous component of degree d of f, K_∞ be the asymptotic cone of K. We say that $\mathrm{OP}(K, f)$ is regular if the solution set of the asymptotic problem $\mathrm{OP}(K_\infty, f_d)$ is bounded, and the problem is non-regular if otherwise. In 1956, Frank and Wolfe proved that if K is polyhedral, f is quadratic and bounded from below over K, then $\mathrm{OP}(K, f)$ has a solution. Several extensions of the Frank-Wolfe theorem for quadratic, cubic, and polynomial optimization problems have been shown.
The present paper gives another Frank-Wolfe type theorem, which says that if $\mathrm{OP}(K, f)$ is regular and f is bounded from below on K then the problem has a solution. Besides, the Eaves theorem provides us another criterion for the solution existence of quadratic optimization problems. This paper introduces an Eaves type theorem for non-regular pseudoconvex optimization problem $\mathrm{OP}(K, f)$, where K is convex.

Under the assumption that the constraint set K is compact and semi-algebraic, some stability and genericity results for polynomial optimization problems have been shown Lee and Pham in 2016. If K is compact, then its asymptotic cone is trivial, i.e., $K_\infty = \{0\}$, it follows that $\mathrm{OP}(K, f)$ obviously satisfies the regularity condition. The present paper considers the case that K is unbounded. Under the regularity condition, we prove several local properties of the solution map of polynomial optimization problems such as local boundedness, upper semicontinuity, and local upper-Hölder stability. We denote by $\mathcal{R} \subset \mathbb{R}[x]$ the set of all polynomials g of degree d such that $\mathrm{OP}(K, g)$ is regular. This set is an open cone in the space $\mathcal{P}_d$ of all polynomials of degree at most d. At the end of this work, we prove that $\mathcal{R}$ is generic in $\mathcal{P}_d$, if K is given by convex polynomials. The paper has been submitted for publication in Optimization Letters.

These works contribute to Task 1.1 and Task 1.3.

2.2 ESR2 - Andrew Ferguson/SU

Currently, we have been writing up the work described in the abstract we submitted to ISSAC in a paper that we aim to submit by the end of November 2020. The paper, "On the computation of asymptotic critical values of polynomial maps and applications" describes progress towards realising a polynomial optimisation which yields an exact representation of the infimum of the input polynomial that can be used in practice. To this end we consider the set of generalised critical values which is the union of the classical critical values and the asymptotic critical values. This is a set containing the so-called bifurcation set of a polynomial. From this set it is possible to compute an exact representation of the infimum of a polynomial as well as decide whether this infimum is reached. To this end, we provide new algorithms to compute part of the union that is the generalised critical values, the asymptotic critical values. Through randomisation we are able to develop the work of Kurdyka, Orro and Simon in their paper "Semialgebraic Sard Theorem for generalised critical values", which as far as we know gives the first geometric characterisation of the asymptotic critical values that allows for their computation. Our new characterisations allow us to dramatically reduce the complexity of computing these values. We also obtain tighter degree bounds on a hypersurface containing the asymptotic critical values of a polynomial mapping. Our implementation of the algorithms we develop in this paper shows how the practical efficiency surpasses the current state-of-the-art algorithms for computing asymptotic critical values by tackling example problems that were previously out of reach.

Furthermore, we have been working in collaboration with Professor Alin Bostan, of the SpecFun team in INRIA, on a seperate branch of research where we aim to give a new understanding of the structure of the Gröbner bases involved in critical point/value computation. This would allow us to give new complexity bounds on the computation of generalised critical values which, as discussed above, begets a strategy for exact polynomial optimisation among other problems in computer algebra such as quantifier elimination. Currently, we have proven results involved with the Hilbert series of generic determinantal ideal s and the staircases they describe. These results allow us to relate the coefficients of the Hilbert series to the structure of the multiplication matrices that arise in the Gröbner basis computations used to compute the critical values of a polynomial. We aim to submit this work to the next ISSAC conference held in Summer 2021.

The reported work contributes to Task 1.1.

2.3 ESR3 - Markus Nordvoll Breivik/UKON

This project is concerned with polynomial optimization problems (POP's) with symmetry. To this end we are looking at real varieties and semi-algebraic sets which are invariant under the action of some group. We refer to these as invariant real varieties and invariant semi-algebraic sets. In particular, we are interested in knowing when such sets are empty, as this can tell us whether our POP is feasible or not. If $S \subset \mathbb{R}^n$ is a semi-algebraic set, invariant under the action of G, then we call a subset $R \subset S$ a test set (of S) if S is empty if and only if R is empty. In the case when G is the symmetric group S_n and our real variety $V \subset \mathbb{R}^n$ is defined by invariant polynomials $f_1, \ldots, f_r$ of degree at most d, $d \geq 2$, Timofte's half-degree principle [Rie12] tells us that the subset R of V consisting of points with at most $\lfloor d/2 \rfloor$ distinct components is a test set for V. We would like to generalize the half-degree principle to other groups, or to strengthen it to give better test sets. In the proof of the half-degree principle, properties of the hyperbolicity sets

$$\mathcal{H}(a_1, \ldots, a_s) = \{z \in \mathbf{R}^n : z_1 = a_1, \ldots, z_s = a_s, T^n - z_1 T^{n-1} + \cdots \pm z_n \text{ is hyperbolic}\}$$

are used. These sets seem to possess stronger properties that those employed in the proof, and we are currently studying them for further exploitation.

The hyperbolicity sets occurring in [Rie12] comes from considering the orbit space of the semi-algebraic sets. For problems involving symmetry, it is a common strategy to study the corresponding problem in the orbit space. A result by Procesi and Schwartz [PS85] gives an explicit description of the orbit space of $\mathbb{R}^n$ as a semi-algebraic set. We are currently looking for analogous descriptions of orbit spaces of invariant subvarieties $V \subset \mathbb{R}^n$, which would be useful for giving criteria for test sets.

Another aspect to consider is the algorithmic complexity of determining whether an invariant semi-algebraic S set is empty. This is related to the (equivariant) Betti numbers of S. The paper [BR16] gives bounds for the equivariant Betti numbers of S in the case when $G = S_n$. We will together with Sebastian Debus (ESR11) and Cordian Riener try to further this work, and as a particular application, bound the Betti numbers of projections of semi-algebraic sets.

Tangential to the problem of determining emptiness for invariant varieties and semi-algebraic sets, are discriminants for invariant homogeneous polynomials. The paper [BK14] presents a factorization of the discriminant of symmetric forms, and we are working to extend this factorization to finite reflection groups in general.

This work contributes to Task 1.2 and Task 3.1, Workpackage 3.

2.4 ESR4 - Alejandro González Nevado/UKON

The spectrahedral relaxation introduced by my advisor M. Schweighofer in his paper [Sch19] shows already interesting properties in terms of simplicity of its computation and good approximation of rigidly convex sets (RCS) of real zero (RZ) polynomials. However, towards a solution of the Generalized Lax Conjecture (GLC), approximations are not enough as we require an exact multiple of the polynomials (by another polynomial having a strictly bigger RCS) to have a monic linear matrix polynomial determinantal representation (MLMPDR). The good properties that this relaxation exhibits make us think that extending this relaxation by allowing more or a better compilation of the information of the RZ polynomial describing the RCS in which we are interested could result in a breakthrough towards a solution of the GLC (at least for small degree). At this moment, we cannot confirm that this will be the case but we found some results that seem to show that certain extensions allowing the interplay of more polynomial information (coming from the coefficients of the original polynomial) could be possible. In particular, we found bigger matrices than those used in the original paper that behave in a nice way in the sense that it can be proved that any RZ polynomial $p \in \mathbb{R}[x]$ produces a(n associated) form $L_{p,d}$ for $d \geq \deg(p)$ whose application to every entry of these new bigger matrices gives as a result initial (at $x = 0$) matrices which are always positive semidefinite (PSD) (positive definite (PD) under mild conditions), which means that a process similar to the one described in the original paper continues giving as a result RZ polynomials under the mentioned mild conditions. The problem is that these new RZ polynomials do not produce relaxations of the RCS we begin with. We discovered this while doing experiments with the software Mathematica (and so we will continue) and now we have theoretical proofs for the first statements in that direction. Another line that this aimed us to pursue is finding better ways of expressing (preferably explicitly) these L forms, which, in the original paper, are primarily implicitly defined although the original paper already provides explicit expressions in terms of the coefficients of the original polynomial for the value over small degree monomials and for the value of every monomial under the availability of an MLMPDR in terms of the traces of the matrix coefficients of the associated MLMP. In our research we already found expressions extending those given in terms of the coefficients of the original polynomial for every monomial developing some combinatorics of multisets, which has provided us with an additional way of researching in the behaviour of the L forms.

In relation to the project developed during the secondment in UNIFI I think that the best option is to provide the reference [Tur20] to the arXiv preprint and to mention that Ettore and I were able to extend with the advise of Giorgio the description of some binary forms of suprageneric rank in terms of multiple root loci and a theorem of Comas-Seiguer about the decomposition of the variety generated by forms of fixed

rank in the subgeneric rank case to the suprageneric rank case. This paper has already been submitted to a journal.
The reported work contributes to Task 1.1, Task 1.2 and Task 1.3.

2.5 ESR5 - Ettore Turatti/UNIFI

Ettore Turatti's project is now in a phasis touching directly Polynomial Optimization. We consider the tensor space given by the tensor product of finitely many real vector spaces of finite dimension. This space may be equipped by the Frobenius metric induced by a Euclidean metric fixed in each space. Inside this space there is the cone variety of decomposable tensors.

A basic problem in optimization setting is to analyze and compute the closest decomposable tensor to a given tensor. This is also called the best rank one approximation of a given tensor.

A way to certificate that a local minimum of the distance function is also a global one is to compute all critical points of the distance function and to check the distance of all of them.

Indeed it is convenient to check all the complex critical points, which are finitely many and their number is known in general theoretically, by formulas given by Fornaess-Sibony and Cartwright-Sturmfels in the symmetric case and by Friedland-Ottaviani in general.

These critical points are called tensor eigenvectors in the symmetric case and singular t-ples in the general case. The set (scheme) of the eigenvectors or singular t-ples of a tensor T is called eigenscheme of T.

A natural problem is to ask if any tensor T can be reconstructed from its eigenscheme. This is false for matrices, since summing to any matrix a scalar multiple of the identity the eigenscheme is not modified. Anyway it is easy to prove that this is the only obstruction to the reconstruction problem, namely the fibers of the map which associates to a matrix T its eigenscheme are given exactly by T+cI where c is a scalar and I the identity matrix.

A recent preprint by Beorchia, Galuppi and Venturello solved this problem for symmetric tensors on a vector space of dimension at most three. The answer is similar to the matrix case for even degree, while for odd degree a strong bijection is proved, namely the tensor can be uniquely reconstructed from its eigenscheme in odd degree.

The first result by Turatti, still unpublished but the preprint is in preparation and at a good point, is that the result of Beorchia, Galuppi and Venturello can be generalized to any number of variables. Turatti proof is different and it uses Representation Theory and Vector Bundles, exactly as in the spirit of the Poema project and according to the training performed until now.

Even more, there are hopes that the result could be extended from the symmetric case (Veronese) to the general case (Segre) and even to the very general spaces spanned by Segre-Veronese varieties, but this is still a project for the future.

The node participates also to an Italian group in Applied Algebraic Geometry that, after a period of lockdown, decided recently to continue online a seminar cycle, see http://web.math.unifi.it/gruppi/algebraic-geometry/AppliedAlgebraicGeometry20202021.html

The reported work contributes to Task 1.3.

2.6 ESR6 - Andries Steenkamp/CWI

We have worked on an optimization problem, dealing with the minimization of certain classes of hypergraph-based polynomials over the standard simplex. This optimization problem is motivated from a question in queuing theory, posed by Cardinaels, Borst and van Leeuwaarden [ECL20], asking to evaluate the performance of redundancy strategies for scheduling tasks on parallel machines. We present here our main results on the relevant polynomial optimization problem.

Given $n, L \in \mathbb{N}$, let E denote the collection of L-subsets of V, the edge set of the complete hypergraph on $V = [n]$. Given an integer $d \geq 2$ we consider the following two polynomials:

$$p_d(x) = \sum_{(e_1,\ldots,e_d)\in E^d} \frac{1}{|e_1 \cup \ldots \cup e_d|} x_{e_1} \cdots x_{e_d}, \tag{1}$$

and

$$f_d(x) = \sum_{(e_1,\ldots,e_d)\in E^d} \prod_{i=1}^{d} \frac{x_{e_i}}{|e_1 \cup \ldots \cup e_i|}. \tag{2}$$

Hence, the coefficients depend on the cardinality of the union of certain tuples of hyperedges. We consider the problem of minimizing these polynomials over the standard simplex:

$$\Delta_{|E|} = \Big\{ x = (x_e)_{e\in E} \in \mathbb{R}^{|E|} : x \geq 0, \sum_{e\in E} x_e = 1 \Big\},$$

whose elements can be seen as probability distributions over the hyperedges. The question posed in [ECL20] is whether the minimum is attained at the barycentre of the simplex. We have been able to give a positive answer to this question for the polynomials p_d and for a special case of the polynomials f_d, the results can be found in [DB20] (submitted for publication).

We now sketch the main steps in our proof. A crucial ingredient in the proof is exploiting the fact that the polynomials enjoy many symmetry properties. The first step is to reduce the problem to showing that the polynomials are convex. Then the next step is expressing their Hessian matrices as matrix polynomials, so that the final task consisted in showing that certain families of well structured matrices are positive semidefinite.

The matrices in these families did enjoy a lot of symmetries that we could exploit to complete the proof. Indeed, after certain reductions, we did end up with matrices lying in the Terwilliger algebra $\mathcal{A}_n$ of the binary Hamming scheme, i.e., with matrices of the form $\sum_{i,j,t\geq 0} x^t_{i,j} D^t_{i,j}$, where $x^t_{i,j}$ are scalars, and $D^t_{i,j}$ is the matrix indexed by subsets of V, with (S,T)-entry 1 if $|S| = i$, $|T| = j$ and $|S \cap T| = t$, and 0 otherwise.

This was a critical insight, which allowed us to invoke a powerful result. Since $\mathcal{A}_n$ is a matrix *-algebra, by the theorem of Artin-Wedderburn [Wed64; GPBK], it can be block-diagonalized, and the explicit block-diagonalization was given by Schrijver [Sch05]. This, combined with an integration trick, enabled us to conclude the proof of positive semidefiniteness of the desired matrix classes, and thus the convexity of the polynomials p_d.

We have not shown f_d to be convex in full generality. This setback is largely due to its Hessian matrix being much more complex now than it was for p_d. Resolving this question for the polynomials f_d may require to exploit different (more complicated) symmetry properties, and the use of more advanced techniques from representation theory.

The reported work contributes to Task 1.1 and Task 2.1, Workpackage 2.

3 Related activities of the network in WP1

3.1 Inria

We focus on the analysis of Moment Matrix (MoM) convex relaxations in polynomial optimization problems and in exploiting their structure for the solution of algebraic problems in real algebraic geometry. In particular, we consider the problem of computing the real radical of an ideal, or the defining ideal of a basic semi-algebraic sets. We are investigating new algorithms for computing generators of these defining ideals, which rely on the knowledge of truncated moment sequences, obtained from MoM relaxations. An article is in preparation on this topic.

3.2 CNRS

Given a compact basic semi-algebraic set, the framework from [LM20] provides a numerical scheme to approximate as closely as desired, any finite number of moments of the Hausdorff measure on the boundary of this set. This framework can be used to approximate interesting quantities like length, surface, or more general integrals on the boundary, as closely as desired from above and below.
The result from [WM20] states a constructive proof that the cone of sums of non-negative circuits (SONC) admits a second-order cone representation. Based on this, a new algorithm is provided to compute SONC decompositions for certain classes of non-negative polynomials via second-order cone programming. Numerical experiments demonstrate the efficiency of this algorithm for polynomials with fairly large degrees and numbers of variables.

One of the similar features shared by SOS/SONC-based frameworks is their intrinsic connections with conic programming: SOS decompositions are computed via semidefinite programming and SONC decompositions via geometric/second-order/linear programming. In both cases, the resulting optimization problems are solved with interior-point algorithms, thus output approximate non-negativity certificates. However, one can still obtain an exact certificate from such output via hybrid numerical-symbolic algorithms when the input polynomial lies in the interior of the SOS/SONC cone. One way is to rely on rounding-projection algorithms adapted to the SOS cone, or alternatively on perturbation-compensation schemes as in [MD20].

This scheme has been used to provide in [DMS20] a computer-assisted approach to ensure that a given continuous or discrete-time polynomial system is asymptotically stable. The resulting framework relies on constructive analysis together with formally certified SOS Lyapunov functions. The crucial steps are formalized within the proof assistant Minlog.

The CNRS POEMA members participated in the workshop "Real algebraic geometry with a view toward hyperbolic programming and free probability" in Oberwolfach, March 1–7, 2020. V. Magron has organized at LAAS the workshop "Brainstorming day on polynomial optimization".

3.3 CWI

Consider the problem of finding the minimum value $f_{\min}$ taken by a polynomial f on a compact set K. Lasserre [Las11] introduced a hierarchy of *upper bounds* that are obtained by searching for an optimal sum-of-squares density of degree $2r$ w.r.t. a given measure μ on K. When the degree r tends to infinity the bounds $f^{(r)}$ converge to $f_{\min}$. Lasserre [Las19] proposed a weaker (but cheaper) hierarchy of bounds, obtained by searching for *univariate* sum-of-squares densities with respect to the push forward measure of μ by f, denoted μ_f, considered on the interval of values taken by f. In [SLbl] we have been able to analyze the quality of the bounds produced by this weaker hierarchy and to show that the error is of the order $\log(r)^2/r^2$. For this we used so-called needle polynomials to get good sum-of-squares densities. This analysis applies in particular when K is a compact semi-algebraic set which is fat (i.e., its interior is dense in K).

3.4 SU

The problem of computing critical points is at the heart of POP since minima of polynomial functions under polynomial constraints are reached at these points. Surprisingly, the complexity of computing algebraic representations of such geometric objects was not fully understood. This can be done by improving the complexity of algorithms solving the class of so-called determinantal systems (composed by maximal minors of some polynomial matrix). In [Hau+21], we prove that isolated points of determinantal systems can be computed in time which is polynomial in the generic number of such solutions by designing a homotopy algorithm dedicated to this class of problems.

In [Lab+20], we restrict our homotopy algorithm to obtain better runtimes adapted to the situation where the entries of the matrix share the same degree per column. This is one component of the algorithm in [Fau+20] which focuses specifically on computing critical points of polynomial maps under polynomial constraints which are all invariant by the action of the symmetric group. This yields an algebraic proof that such points can be computed in time polynomial in the number of variables on families where the input degrees are fixed. This algorithm has also subexponential complexity for some families where e.g. the input degrees are smaller than the square root of the number of variables.

Moment matrices appear ubiquituously in algorithms of effective real algebraic geometry, in particular the co-called Hermite matrices which are classically used for real root counting. We extend slightly these constructions and obtain algorithms for classifying the real roots of polynomial systems with parameters in [LS20]. The output formula obtained here have asymptotically optimal degree.

All of this is used in [LSW20] for computing the isolated real points of complex hypersurfaces in view of applications in mechanism design and in [CSS20] and [Tru+20] (jointly with CNRS) for applications of POP in robotics. Last but not least, we participated in the workshop "Real algebraic geometry with a view toward hyperbolic programming and free probability" in Oberwolfach, March 1–7, 2020.

3.5 UKON

The Konstanz POEMA members participated in the workshop "Real algebraic geometry with a view toward hyperbolic programming and free probability" in Oberwolfach, March 1–7, 2020.

Claus Scheiderer participated in the online meeting of the German Mathematical Society in Chemnitz, September 14–17. He carried out a "Research in Pairs" stay in Oberwolfach, jointly with Gennadiy Averkov, 11 - 17 October. A follow-up stay is planned for February 2021.

A considerable number of planned invitations for 2020 were cancelled because of the pandemic.

3.6 UNIFI

Giorgio Ottaviani gave some online talks on optimization of distance function in tensor spaces, at SISSA in Trieste in April 2020, at University of Insubria in November 2020. Giorgio Ottaviani completed a preprint, joint with Zahra Shahidi, on tensors with eigenvectors on a fixed linear subspace, arXiv:2010.03843. He also worked with L. Sodomaco and E. Ventura on Asymptotics of degrees and ED degrees of Segre products, arXiv:2008.11670. He completed the lecture notes of a course given at Berlin TU in 2019, with the collaboration of P. Reichenbach, on Tensor Rank and Complexity [OR]. Several planned activities were canceled due to the pandemic emergency. The UNIFI node participates also to an Italian group in Applied Algebraic Geometry that, after a period of lockdown, decided recently to continue online a seminar cycle, see http://web.math.unifi.it/gruppi/algebraic-geometry/AppliedAlgebraicGeometry20202021.html

4 Conclusion

The covid-19 outbreak has been difficult for everyone. It impacted (and continues to impact) a lot of young people and our ESR had to face this difficult situation while they were starting their PhD in a foreign country, far from their home and family.

Our greatest satisfaction is to see that despite this, they already succeeded to yield some nice accomplishments on all the tasks of this workpackage. All of them have either already submitted a paper or are close to this point. We expect for the next year more results and hope to initiate some collaborations between the POEMA nodes. Some first software packages may also be finished.

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