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Deliverable D4.1

Research progress on applications of polynomial optimization

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1 Introduction

The objectives of this workpackage are:

- to leverage state-of-the-art techniques from polynomial optimization, semi-algebraic optimization, and algebraic tools for moment problems to tackle challenging decision optimization problems arising from industrial applications.
- to demonstrate potential of polynomial optimization to find better solutions to industrial decision optimization problems.
- to demonstrate potential of polynomial optimization to solve novel practical decision optimization problems.
- to demonstrate scalability of polynomial optimization tools.

The workpackage is divided in the following tasks:

Task 4.1: Demonstrate potential of binary polynomial optimization to the solution of truss topology optimization problems with binary or integer variables (led by UoB). In truss topology optimization (TTO) we want to design a pin-jointed mechanical framework called truss consisting of slender bars connected at flexible joints. The design variables are the volumes of the bars and the goal is to find, for specific boundary conditions and loads, a lightest truss with prescribed stiffness. The basic TTO problem can be formulated as a continuous polynomial optimization problem. The challenging technological constraints may require that the bar volumes can only take values from a finite set, typically an integer value. In the simplest scenario, we may only require binary values of the variables. The objective of this task is to build a library of TTO models and problems, in order to test the performance of binary polynomial optimization solvers developed within POEMA. TTO problems are suitable for benchmarking, as they are easily scalable from very small to very large dimension. They also offer plethora of models, from LP to nonlinear SDP, with various equivalent formulations.

Task 4.2: Demonstrate potential of binary polynomial optimization for novel approaches to operating water and power systems and in urban traffic management (led by UTIL). A large class of urban network problems exhibits a common structure where in addition to the nonlinear constraints and the continuous decisions that model the system's dynamics there are also binary on/off decisions. Examples of such challenging problems include, valve placement, switch gear operations, pump scheduling, and traffic management. In general, these problems can be formulated as binary polynomial programs which continues to be a challenging area of research. The objective of this task is to develop a framework to solve binary polynomial programs based on the integration of semidefinite programming and cut generation techniques and applying the proposed methodology to urban distribution networks. In particular, this task focuses on exploring the possibility and the advantages of generating polynomial valid inequalities, devising nonlinear branching rules, and designing efficient heuristics.

Task 4.3: Demonstrate potential of continuous polynomial optimization for fluid dynamics problems arising from oceanography and environmental monitoring (led by CNRS). Navier-Stokes equations are in the core of numerical simulations in oceanography. These simulations, in turn, serve a basis for different environmental monitoring services and are widely applied in the framework of IBM's Smart Planet program. 3D acoustic and elastic wave equations are widely applied in industry for oil and gas exploration. Darcy equations are widely used in geophysics for describing the flow of groundwater through an aquifer. Remarkably, the discretisation of the mentioned equations generates polynomial ordinary differential equations defining evolution of parameters, which define the approximation. The aim of this task is to develop scalable global continuous optimization methods based on moment approach and apply them to industrial inverse problems.

Task 4.4: Demonstrate scalability of polynomial optimization on power network optimization problems (led by ARTELYS). A common feature of several challenging problems in resource distribution networks is that they are naturally cast as large-scale polynomial optimization problems, which inherit the sparsity structure of the underlying distribution network. A major advance in numerically efficient methods for large-scale polynomial optimization problems derives from exploiting problem structure such as sparsity and symmetry. Another common real-world challenge is polynomial optimization in the presence of data uncertainty. The aim of this task is to demonstrate the scalability of polynomial optimization approaches and to develop robust optimization frameworks by successfully tackling industrial instances of resource network optimization problems, e.g. pressure management in water distribution networks or optimal power flow in power systems.

2 Status of research performed by ESRs

2.1 ESR7 - Luis Felipe Vargas Beltran/CWI

Quadratic programming is a special class of polynomial optimization problems in which the objective function is a quadratic polynomial and the constraints are linear. Another special class is standard quadratic Programming, which is a quadratic program in which the feasible region is the simplex. That is, a problem of the form:

$$\min \quad x^T M x \quad \text{subject to} \quad x \geq 0,\ \sum_{i=1}^{n} x_i = 1, \tag{1}$$

where M is a symmetric matrix.

Many problems can be formulated as standard quadratic programs, such as the maximum stable set problem via the Motzkin-Straus formulation and portfolio optimization. This is turn implies that standard quadratic optimization is NP-hard. Therefore one needs to use efficient approximation methods to get tractable bounds.

In our project, we investigate the performance of the Lasserre hierarchy applied to standard quadratic programs. In particular, we are interested in identifying for which class of matrices M we have finite convergence of the Lasserre hierarchy. For this, we exploit a result of Nie [Nie14] and Marshall [Mar08] which allows us to prove finite convergence under some optimality conditions on nonlinear programming. As a consequence, we can show finite convergence of the Lasserre hierarchy for problem (1) when the matrix M has diagonal entries equal to 1 and its off-diagonal entries are either 0 or greater than 1. A particular application of this result is by considering the following ϵ-perturbed Motzkin-Straus formulation for computing the stability number of a graph:

$$\frac{1}{\alpha(G)} = \min x^T((1+\epsilon)A_G + I)x \quad \text{subject to} \quad x \geq 0,\ \sum_{i=1}^{n} x_i = 0. \tag{M-S-ϵ}$$

Here we select $\epsilon > 0$. Then, the corresponding Lasserre hierarchy has finite convergence.

A restriction of the result of Nie and Marshall is that it can only be applied to the case in which the problem has finitely many global minimizers. This condition motivates the restriction on the class of matrices M in (1) (and those appearing in ((M-S-ϵ))).

On the other hand, recent work of Ahmadi and Zhang [AZ19] focused our attention to complexity questions in polynomial optimization. In particular, they proved that it is strongly NP-hard to decide if a polynomial optimization problem has a local minimum. Building up on our understanding about the number of global minimizers of problems (1) and (M-S-ϵ), we proved that, unless P=NP, there is no polynomial time algorithm to decide whether a standard quadratic program has finitely many global minimizers. To prove this we exploit perturbations of the Motzkin-Straus formulation and obtain complexity results from classic combinatorial problems. One of the goals of the project is to analyze more complexity questions in polynomial optimization problems by exploiting instances related with hard combinatorial problems.

This work is in relation with Task 4.2.

2.2 ESR8 - Felix Kirschner/UvT

In portfolio optimization a typical question to ask is what is the optimal distribution of some budget B among n assets such that some objective such as expected return is maximized. One is looking for the optimal proportion $x_i \in [0,1]$ of the budget B to be invested in asset $i \in [n]$. The optimal portfolio is thus a point $x \in \Delta_n = \left\{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\right\}$. We analyzed a RLT- and a Lasserre-type hierarchy for polynomial optimization problems over the simplex and proved a rate of convergence of $O(1/r)$ for both

hierarchies under some assumptions. The convergence rate was also found to be tight. Both hierarchies were implemented in Julia.
The second project we started working on is rooted in approximation theory. Given a smooth function f defined on $[-1,1]^n$ the goal is to approximate f by a sequence of polynomials of increasing degree such that the sequence converges to f uniformly on $[-1,1]^n$. Formally, we consider the sequence of positive, linear approximation operators:

$$\mathcal{K}^{(r)}(f)(x) = \int_{[-1,1]^n} f(y)K_r(x,y)d\mu(y),$$

where K_r is a approximation kernel and μ is a probability measure on $[-1,1]^n$. The kernels that qualify themselves as suitable for our task must fulfill certain properties. We aim to gain a deeper understanding of the *optimal* kernels, in the sense that they lead to the quickest uniform convergence. We emphasize that these kernels may also be utilized to bound the convergence rate of the Lasserre hierarchy of lower bounds over the hypercube under some assumptions. In our analysis we did some numerical experimenting, where we implemented a symmetry reduction for the underlying SDP. The symmetry that was exploited was induced by the group $S_2 \times S_n$.

This work is in relation with Task 4.2.

2.3 ESR9 - Lorenzo Baldi/Inria

The ESR9 Lorenzo Baldi is involved in the development of MomentTools.lj, a Julia package for Polynomial Optimization on moments, and of MultivariateSeries.jl, a Julia package for manipulation of series indexed by monomial exponents and polynomial-exponential decomposition. He is also contributing to the POEMA database of polynomial optimization problems.

He exchanged with E. Fuentes (ESR 15) and his advisor about an optimal power flow problem.

2.4 ESR13 - Soodeh Habibi/UoB

The ESR13 has been following research on the applications of polynomial optimization to problems of structural optimizations. Two research directions have been followed.

The first one concerns formulations of the truss topology optimization problem [Koč17] as a semidefinite optimization problem with low-rank or approximate low-rank solutions. We have shown that for problems with non-zero lower bound on the truss member volumes, the solution of the SDP problem is exactly of rank one, while for zero lower bounds the rank may be higher and depends on a particular truss configuration, boundary conditions and loads. The same study was performed for truss topology problems with constraint on free vibrations, leading to SDP with two large-scale matrix variables. We have generated a number of test problems, ranging from very small to very large dimension and uploaded these problems in POEMA database [KM20].

The second research direction focuses on algorithms for SDP problems with (very) low-rank solutions, as those described above. We have developed an interior point method with inexact solution of the Schur complement (linear) system. The system is solved by the preconditioned conjugate gradient method, whereas the newly developed preconditioner utilizes the low-rank property of the solution. Preliminary results show that, for the truss topology problems from POEMA database, the developed Matlab software is significantly more efficient than state-of-the-art commercial software for SDP. This work was performed in close collaboration with ESR12 (FAU).

This work contributes to Task 4.1.

2.5 ESR14 - Corbinian Schlosser/LAAS-CNRS

The ESR14 has been following research on the applications of polynomial optimization to dynamical systems. Two research directions are followed.

The first one tackles new, previously unsolved, problems using polynomial optimization. Currently, the main result in this direction is the characterization of the global attractor as the infinite-dimensional linear programming problem in the space of Borel measures whose dual, when approximated using sum-of-squares of polynomials, provides outer approximations to the global attractor with guaranteed convergence [SK20a].

The second research direction tackles structure exploitation in the polynomial optimization problems arising from dynamical systems. The main result there is a method for sparsity exploitation with a guaranteed convergence of the semidefinite programming hierarchy [SK20b]. Contrary to the case of static polynomial optimization, exploiting sparsity in the context of dynamical systems without forgoing convergence guarantees has been an open question and this work is the first one to give a positive answer, to the best of our knowledge.

Future work will follow these two research directions. For the former one, we envisage tackling, for example, nonlinear partial differential equations. For the latter one, we plan to exploit other types of structure found in dynamical systems.

This work contributes to Task 4.3.

2.6 ESR15 - Edgar Fuentes/Artelys

The efficiency of Polynomial approximations in order to build strong relaxation of the Optimal Power Flow problem (OPF) has been investigated.

The ESR has first investigated existing relaxations (including SDP relaxations) as well as their drawbacks. He has then focused on the possible exactness of second order moments relaxations and their computational tractability.

Given the high computational burden of solving such relaxations, he has been working on approaches allowing to locally increase the order of the relaxations in order to strengthen particular constraints, improve the accuracy of the relaxations and make them tractable by (i) having few higher order moments and (ii) creating and exploiting a sparse structure of the constraints matrix. First results demonstrate that these approaches indeed allow strengthening the relaxations while lowering the computational burden. Some future work will strive towards building a complete and tractable relaxation scheme using this approach.

Besides the computational aspects, one of the main practical difficulties of using such relaxations is to use their results in order to compute feasible solution(s) whenever the relaxation is not exact. Projecting such solutions onto the feasible space and generating good feasible solutions for the original problem is going to be one of the main research topics in coming months.

This work contributes to Task 4.4.

3 Related activities of the network in WP4

3.1 Inria

With the aim of exploiting the analysis of moment structure for polynomial optimisation developed in the other packages, we develop a dedicated package for Polynomial and Moment optimization in the language Julia, called `MomentTools.jl` and experimented it on few simple problems.

We contributed to Polynomial Moment Optimisation Database developed in POEMA, by specifying the storage format [KM20] and by developing a Julia package `PMO.jl` for the exploitation and management of the database.

3.2 CNRS

The CNRS POEMA members have developed several frameworks to exploit sparsity arising in the input of polynomial optimization problems.

Several algorithms have been designed for specific problems with few correlations between the variables of the input data, for instance deep networks in [Che+20], positive definite forms in [MML20b] or eigenvalue/trace optimization of non-commutative polynomials in [KMP19].

Another framework has been developed to exploit sparsity of polynomial optimization problems by considering sparsity patterns based on terms, rather than variables, yielding an alternative sparse variant of Lasserre's hierarchy.

The resulting *term-sparsity* SOS (TSSOS) hierarchy has been developed in [WML20b; WML20a], combined with correlative sparsity in [Wan+20], and extended to the non-commutative case in [WM20]. Further applications include fast computation of upper bounds for the joint spectral radius for control-systems with deadline misses constraints [WMM20; Vre+20].

Further improvements have been made to improve the scalability of the semidefinite relaxations arising from polynomial optimization.

In [MML20a], it is proved that these relaxations satisfy the constant trace property for the involved matrices. This property is exploited to avoid solving the relaxations via interior-point methods and rather use ad-hoc spectral methods that minimize the largest eigenvalue of a matrix pencil. Efficiency and robustness has been tested against several equality constrained problems on a sphere as well as on a sample of randomly generated quadratically constrained quadratic problems.

Another research line focuses on optimizing trace polynomials, i.e., polynomials in non-commuting variables and traces of their products. In [KMV20], a novel Positivstellensatz certifying positivity of trace polynomials subject to trace constraints is presented, and a hierarchy of semidefinite relaxations converging monotonically to the optimum of a trace polynomial subject to tracial constraints is provided. The Gelfand-Naimark-Segal construction is applied to extract optimizers of the trace optimization problem if flatness and extremality conditions are satisfied.

V. Magron has organized at LAAS the workshop "Brainstorming day on polynomial optimization".

3.3 Artelys

ESR and advisors have been involved in POEMA network activities (trainings, workshops, etc). Edgar Fuentes (ESR15) has also contributed to the creation of the database of polynomial optimization problems (part of the POEMA project) and testing of the corresponding Julia packages.

3.4 UoB

Starting with the secondment of ESR12 in UoB (Summer 2020) we have organized regular bi-weekly meetings with the FAU team. A particular focus of these meetings was to identify common structures in two algorithms for semidefinite optimization, interior point methods and primal-dual methods based on Augmented Lagrangians. It turned out that the linear systems which have to be solved at the core of both methods have a mathematically equivalent structure. Thus it seems possible to develop iterative methods and, in particular, preconditioners, which can be applied in the framework of both solution concepts. These methods have been tested by means of problems from truss topology design. Part of this collaboration has been finalized during the secondment of ESR13 in FAU (Autumn 2020) and will be published in a joint article (to be finished and submitted in early 2021).

4 Conclusion

This document presented activities related to applications of polynomial optimization during the first two years of the project, as well as the results obtained in these activities.
The activities in WP4 follow the plan, despite unexpected problems related to COVID-19, in particular some of the postponed secondments. The workpackage is developing at large in a very positive way.

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