# Polynomial optimal control: references, exercises, questions and answers

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## 1 References

The moment-SOS hierarchy was first applied to polynomial optimal control (POC) in [15]. It fits the framework of the generalized problem of moments [14, 13]. The lecture follows closely the presentation of [10]. See also [9] for sketchy lecture notes.

The moment-SOS hierarchy can be seen as an alternative to standard numerical methods for POC. It is a global method generating lower bounds on the value function, while local methods based e.g. on the Pontryagin Maximum Principle (necessary conditions of optimality) or discretization (local optimization algorithms) generate upper bounds. If the lower and upper bounds coincide, then on the one hand, it is not necessary to go deeper in the hierarchy, and on the other hand, it is not necessary to try other initial conditions or discretize further in the local methods. This strategy was followed in [1] for solving a POC problem in data science.

The moment-SOS hierarchy is a global method bearing similarities with the Hamilton-Jacobi-Bellmann (HJB) approach to POC which consists of solving a non-linear PDE, see e.g. [7, Section 10.3] or [20, 5]. It can be interpreted as a convex relaxation of the HJB PDE.

The Brockett integrator of nonlinear systems control is used as a numerical example for the application of the moment-SOS hierarchy to POC in [15]. It is also known (up to a change of coordinates) as the unicycle or Dubins system, one of the simplest instance of a non-holonomic system in robotics, see e.g. [6] for the connection. It was studied thoroughly in [19], see also [18].

Linear formulations of optimal control problems (on ODEs and PDEs) are classical, and be tracked back to the work by L. C. Young, Filippov, Warga or Gamkrelidze. For more details see e.g. [8, Part III]. The main idea is that in calculus of variations or optimal control, the optima are not attained, i.e. the problems typically do not have solutions when formulated

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in smooth functional spaces (e.g. continuous functions of time, or measurable functions of time). Once formulated in a dual space, e.g. a space of measures, optima are generally attained [16].

Relaxed controls, also called Young measures in the calculus of variations literature, are designed to capture oscillations at the limit, when the frequency tends to infinity. Some POC problems feature a different limit behavior, namely a concentration of the control signal. This is the case for impulsive POC problems arising in space engineering, for which the moment-SOS hierarchy has been adapted [2, 3]. DiPerna-Majda measures, an extension of the Young measures, can be used to deal with the simultaneous presence of oscillations and concentrations, also with the moment-SOS hierarchy [4, 11].

More recently, efforts were dedicated to applying the moment-SOS hierarchy for analyzing and controlling nonlinear PDEs [12, 17].

# 2 Exercises

## 2.1 Exercise 3.1

### 2.1.1 Statement

Relaxed controls capture limit behavior such as e.g. oscillations

$$\lim_{r\to\infty}\int_0^1 v(u_{rt})dt = \int_0^1\int_U v(u)\omega_t(du)dt, \quad \forall v \in C(U)$$

What is the limit $\omega_t(du)$ for $u_{rt} = \cos(2\pi rt)$, $r = 1, 2, \ldots$ ?

### 2.1.2 Solution

To be completed.

## 2.2 Exercise 3.2

### 2.2.1 Statement

The classical Bolza problem

$$\begin{aligned} v^* \;=\; \inf_u \;& \int_0^1 (x_t^2 + (u_t^2 - 1)^2)dt \\ & \dot{x}_t = u_t,\ x_0 = 0 \\ & x_t \in [-1, 1],\ u_t \in [-1, 1]\ \forall t \in [0, 1] \end{aligned}$$

is relaxed to

$$\begin{array}{rcl} v_R^* & = & \inf_\omega \quad \int_0^1 \int_U (x_t^2 + (u^2-1)^2)\, \omega_t(du)dt \\ & & \dot{x}_t = \int_U u\, \omega_t(du),\ x_0 = 0 \\ & & x_t \in [-1,1],\ \omega_t \in \mathscr{P}([-1,1])\ \forall t \in [0,1] \end{array}$$

where $\mathscr{P}([-1,1])$ is the set of probability measures on $[-1,1]$. Prove that there is no relaxation gap: $v^* = v_R^*$ and that the relaxed infimum is attained at $\omega_t^* = \frac{1}{2}(\delta_{-1} + \delta_{+1})$.

### 2.2.2 Solution

To be completed.

## 2.3 Exercise 3.3

### 2.3.1 Statement

Prove that the measure LP

$$\begin{array}{rcll} p^*(t_0, x_0) & := & \min_{\mu,\mu_T} & \int l\mu + \int l_T \mu_T \\ & & \text{s.t.} & \frac{\partial\mu}{\partial t} + \operatorname{div}(f\mu) + \mu_T = \delta_{t_0}\delta_{x_0} \\ & & & \mu \in C([t_0,T] \times X \times U)'_+,\ \ \mu_T \in C(\{T\} \times X_T)'_+ \end{array}$$

has a dual LP

$$\begin{array}{rcll} d^*(t_0, x_0) & := & \sup_v & v(t_0, x_0) \\ & & \text{s.t.} & l + \frac{\partial v}{\partial t} + \operatorname{grad} v \cdot f \in C([t_0,T] \times X \times U)_+ \\ & & & l_T - v(T,.) \in C(\{T\} \times X_T)_+ \end{array}$$

on functions $v \in C^1([t_0, T] \times X)$ and that there is no duality gap: $p^* = d^*$.

### 2.3.2 Solution

To be completed.

## 2.4 Exercise 3.4

### 2.4.1 Statement

In the dual LP

$$\begin{array}{ll} \sup_v & v(t_0, x_0) \\ \text{s.t.} & l + \frac{\partial v}{\partial t} + \operatorname{grad} v \cdot f \in C([t_0,T] \times X \times U)_+ \\ & l_T - v(T,.) \in C(\{T\} \times X_T)_+ \end{array}$$

by combining the dual inequalities evaluated on an admissible trajectory, prove that for any admissible $v$ it holds $v^* \geq v$ on $[t_0, T] \times X$.

### 2.4.2 Solution

To be completed.

# 3 Questions and answers

To be completed.

# References


[1] B. Bonnard, M. Claeys, O. Cots, P. Martinon. Geometric and numerical methods in the contrast imaging problem in nuclear magnetic resonance. Acta Appl. Math. 135(1):5-45, 2015.

[2] M. Claeys, D. Arzelier, D. Henrion, J. B. Lasserre. Measures and LMI for impulsive optimal control with applications to space rendezvous problems. American Control Conference, 2012.

[3] M. Claeys, D. Arzelier, D. Henrion, J. B. Lasserre. Moment LMI approach to LTV impulsive control. IEEE Conference on Decision and Control, 2013.

[4] M. Claeys, D. Henrion, M. Kružík. Semidefinite relaxations for optimal control problems with oscillation and concentration effects. ESAIM: Control, Optimisation and Calculus of Variations, 23:95-117, 2017.

[5] F. Clarke. Functional analysis, calculus of variations and optimal control. Springer, 2013.

[6] D. DeVon, T. Bretl. Kinematic and dynamic control of a wheeled mobile robot IEEE/RSJ International Conference on Intelligent Robots and Systems, 2007.

[7] L. C. Evans. Partial differential equations. American Math. Society, 1998.

[8] H. O. Fattorini. Infinite dimensional optimization and control theory. Cambridge University Press, 1999.

[9] D. Henrion. Optimization on linear matrix inequalities for polynomial systems control. Les cours du C.I.R.M. 3(1):1-44, 2013.

[10] D. Henrion, E. Pauwels. Linear conic optimization for nonlinear optimal control. In S. Ahmed, M. Anjos, T. Terlaky (Editors). Advances and Trends in Optimization with Engineering Applications. SIAM, 2017.

[11] D. Henrion, M. Kružík, T. Weisser. Optimal control problems with oscillations, concentrations and discontinuities. Automatica 103:159-165, 2019.

[12] M. Korda, D. Henrion, J. B. Lasserre. Moments and convex optimization for analysis and control of nonlinear partial differential equations. `arXiv:1804.07565`, 2018.

[13] M. Korda, D. Henrion, J. B. Lasserre. The moment-SOS hierarchy. World Scientific, 2020.

[14] J. B. Lasserre. Moments, positive polynomials and their applications. Imperial College Press, 2010.

[15] J. B. Lasserre, D. Henrion, C. Prieur, E. Trélat. Nonlinear optimal control via occupation measures and LMI relaxations. SIAM J. on Control and Optimization, 47:1643-1666, 2008.

[16] D. G. Luenberger. Optimization by vector space methods. Wiley, 1969.

[17] S. Marx, T. Weisser, D. Henrion, J. B. Lasserre. A moment approach for entropy solutions to nonlinear hyperbolic PDEs. Mathematical Control and Related Fields, 10(1):113-140, 2020.

[18] C. Prieur, E. Trélat. Robust optimal stabilization of the Brockett integrator via a hybrid feedback. Mathematics of Control, Signals and Systems 17(3):201216, 2005.

[19] P. Souères, J. P. Laumond. Shortest paths synthesis for a car-like robot. IEEE Trans. Automatic Control 41(5):672-688, 1996.

[20] E. Trélat. Contrôle optimal : théorie et applications. Vuibert, 2005.