Exercises to “Symmetries in POPs”

15. July 2020

Exercise 1 *Let* $\mathbf{K}$ *a field and* G *be a finite group acting linearly on* $\mathbf{K}^n$*. Show that the set of invariant polynomials* $\mathbf{K}[X_1, \ldots, X_n]^G$ *is a ring.*

Exercise 2 *Let* $\mathbf{K}$ *a field and* G *be a finite group acting linearly on* $\mathbf{K}^n$ *and define for each* $f \in \mathbf{K}[X_1, \ldots, X_n]$ *the operator* $R_G(f) := \frac{1}{|G|} \sum_{g \in G} f^g$*. Show the following properties:*

1. R_G *is a* $\mathbf{K}[X]^G$*- linear map.*
2. *For* $f \in \mathbf{K}[X]$ *we have* $R_G(f) \in \mathbf{K}[X]^G$*.*
3. R_G *is the identity map on* $\mathbf{K}[X]^G$ *i.e.,* $R_G(f) = f$ *for all* $f \in \mathbf{K}[X]^G$*.*

Exercise 3 *Let* $f \in \mathbf{R}[X, Y]$ *be a symmetric polynomial of degree 4. Try to write down the conditions that* f *is a sum of squares polynomial using the Theorem presented in the lecture and compare it the the SDP you would obtain if you did not consider the symmetry.*

Exercise 4 *Let* G *be a finite group and* $f_1, f_2 \in \mathbf{R}[X_1, \ldots, X_n]$ *such that that both polynomials are contained in different isotypic components of* $\mathbf{R}[X_1, \ldots, X_n]$*. Use Schur's lemma to conclude that* $R_G(f_1 f_2) = 0$