Selected Topics on Moment Problems. 1

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Contents

First there were numbers

Let $x_0 > x_1 > 0$ be integers. Euclid division:

$$x_0 = b_0 x_1 + x_2$$

$$x_1 = b_1 x_2 + x_3$$

$$\vdots$$

$$x_{n-1} = b_{n-1} x_n$$

with G.C.D. $x_n = (x_0, x_1)$.

Divide and repeat

$$\frac{x_{k-1}}{x_k} = b_{k-1} + \frac{1}{x_k/x_{k+1}}:$$

$$x_0/x_1 = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \ldots + \frac{1}{b_{n-1}}}} =$$

$$b_0 + \mathbb{K}_{k=1}^{n-1}(\frac{1}{b_k}).$$

Irrationality criteria: the continued fraction does not stop

Hipassus of Metapontum (500 BC): The diagonal x_0 of the square of side x_1 satisfies (via an ingenious geometric recurrence)

$$x_0/x_1 = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}.$$

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More general (Bombelli method, approx. 1560) for N positive integer, not a perfect square:

$$N = a^2 + r, \ \sqrt{a^2 + r} = a + x$$

yields:

$$x = \frac{r}{2a + x},$$

hence

$$\sqrt{N} = a + \frac{r}{2a + \frac{r}{2a+\frac{r}{2a+\dots}}}.$$

The real numbers

For sequences of non-negative integers $b = (b_n)_{n=0}^{J}$, with J finite or not, consider $\mathcal{Z}$ the union of domains

$$\mathcal{D}(b) = \mathbb{Z}_+, \quad b_n > 0, \quad n > 0,$$

or

$$\mathcal{D}(b) = [0, J] \cap \mathbb{Z}_+, \quad J > 0, (b_n > 0, \ n > 0), \ b_J \geq 2,$$

or

$$\mathcal{D}(b) = \{0\}.$$

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Theorem. The mapping

$$\mathcal{Z} \longrightarrow \mathbb{R}, \quad b \mapsto b_0 + \mathbb{K}_{k=1}^{J}(\frac{1}{b_k})$$

is bijective, and a *homeomorphism* from $\mathcal{Z}$ endowed with pointwise convergence.

Algebra of continued fractions

Recurrence, with $a_j \neq 0$:

$$x_0 = b_0 x_1 + a_1 x_2$$

$$x_1 = b_1 x_2 + a_2 x_3$$

$$\vdots$$

$$x_{n-1} = b_{n-1} x_n + a_n x_{n+1}$$

$$\vdots$$

has partial fractions (no cancellation):

$$\frac{P_n}{Q_n} = b_0 + \mathbb{K}_{k=1}^{n}\left(\frac{a_k}{b_k}\right).$$

Main Theorem: Wallis 1656, Brouncker 1655, Euler 1748

The formal continued fraction, with initial data:

$$P_{-1} = 1,\ P_0 = 0,\ Q_{-1} = 0,\ Q_0 = 1$$

implies

$$P_n = b_n P_{n-1} + a_n P_{n-2},$$

$$Q_n = b_n Q_{n-1} + a_n Q_{n-2},$$

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 a_2 \dots a_n,$$

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{\xi_{n+1}}}}}$$

yields

$$\xi = \frac{\xi_{n+1} P_n + a_{n-1} P_{n-1}}{\xi_{n+1} Q_n + a_{n-1} Q_{n-1}}$$

Enters Positivity

Assume all $a_j, b_j > 0$. Then

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = (-1)^{n-1} \frac{a_1 a_2 \ldots a_n}{Q_n Q_{n-1}}$$

therefore

$$\frac{P_0}{Q_0} < \ldots \frac{P_{2k}}{Q_{2k}} < \frac{P_{2k+1}}{Q_{2k+1}} < \ldots < \frac{P_1}{Q_1}.$$

Quadratic irrationals

Periodic continued fraction implies (Euler)

$$\xi = \frac{\xi P_d + a_{d-1} P_{d-1}}{\xi Q_d + a_{d-1} Q_{d-1}}$$

and vice-versa (Lagrange, Galois).
Moebius transform adaptation to mixed period.

Fermat problem

Solve in integers the equation

$$x^2 = 1 + y^2 D,$$

where $D > 0$ is an integer.
If (x_1, y_1) is a solution, then

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$$

is also a solution. Indeed:

$$x_n^2 - y_n^2 D = (x_1^2 - y_1^2 D)^n = 1.$$

Lurking Continued Fraction:

$$x_{n+1} = (2x_1)x_n - x_{n-1}, \quad x_0 = 1,$$

$$y_{n+1} = (2x_1)y_n - y_{n-1}, \; y_0 = 0$$

derived from

$$x_{n+1} + y_{n+1}\sqrt{D} = (x_n + y_n\sqrt{D})(x_1 + y_1\sqrt{D}).$$

Hence:

$$\frac{1}{\sqrt{D}} = \frac{y_1}{x_1} + \mathbb{K}_{k=1}^{\infty}\left(\frac{1}{-2x_1}\right).$$

These are *all* solutions (Lagrange).

Best rational approximation

For an irrational number ξ, the canonical continued fraction obtained by taking integral parts and inversion yields

$$\xi - \frac{P_n}{Q_n} = \frac{\xi_{n+1} P_n + a_{n-1} P_{n-1}}{\xi_{n+1} Q_n + a_{n-1} Q_{n-1}} - \frac{P_n}{Q_n} =$$

$$\frac{1}{Q_n^2 \left(\frac{Q_{n+1}}{Q_n} + \frac{1}{\xi_{n+2}} \right)}.$$

Best Lagrange Approximation

For ξ irrational: every convergent P/Q satisfies

$$|Q\xi - P| < |q\xi - p|$$

for any $p/q \neq P/Q$ and $1 \leq q \leq Q$. And vice-versa.
Moreover, if

$$|\frac{p}{q} - \xi| < \frac{1}{2q^2}$$

then p/q is a convergent of ξ (Legendre).

Bernoulli sequences

Let θ, δ be real numbers and define

$$[(n+1)\theta + \delta] - [n\theta + \delta] = r_n(\theta, \delta).$$

The sequence (r_n) is periodic if and only if θ is rational.

See A. Markov Master Thesis (1879-1880) devoted to Lagrange spectra.

Analytic Theory

Markov's Paradox

Solve

$$z^2 - 2z - 1 = 0$$

Equivalently

$$z = 2 + \frac{1}{z}.$$

The solution should be

$$\xi = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}},$$

that is $\xi = 1 + \sqrt{2}$, because all entries are positive.

Where is the other root $1 - \sqrt{2}$?

The approximants

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2 + \ldots + \frac{1}{2 + 1 - \sqrt{2}}}}}$$

do not converge.

Koch divergence test

Assume $b_n \in \mathbb{C}$ and $\sum_n |b_n| < \infty$.. Then the approximants of $\mathbb{K}_1^\infty(\frac{1}{b_n})$ satisfy:

$$\lim_n P_{2n} = P, \quad \lim_n P_{2n+1} = P',$$

$$\lim_n Q_{2n} = Q, \quad \lim_n Q_{2n+1} = Q',$$

satisfy

$$P'Q - PQ' = 1.$$

Hence clear divergence.

Seidel convergence test

Assume all $b_j > 0$. Then

$$\mathbb{K}_1^\infty(\frac{1}{b_n})$$

converges if and only if

$$\sum_n b_n = \infty.$$

The eternal quest: π

From Wallis:

$$\frac{2}{\pi} = \prod_{j=1}^{\infty} \frac{(2j-1)(2j+1)}{(2j)^2}.$$

and Brouncker:

$$\frac{4}{\pi} = 1 + \mathbb{K}_1^{\infty}\left(\frac{(2n-1)^2}{2}\right).$$

straight to the origins of the analytic theory of continued fractions.

Main idea, derived from Wallis infinite product. Consider a function $b(s) > s$ subject to:

$$b(s)b(s+2) = (s+1)^2.$$

And note:

$$b(1) = \frac{2^2}{b(3)} = \frac{2^2}{4^2} b(5) = \frac{2^2}{4^2} \frac{6^2}{b(7)} = \ldots =$$

$$\frac{2^2}{4^2} \frac{6^2}{8^2} \frac{10^2}{12^2} \cdots \frac{(4n-2)^2}{(4n)^2} b(4n+1) = \frac{1^2}{2^2} \frac{3^2}{4^2} \frac{5^2}{6^2} \cdots \frac{(2n-1)^2}{(2n)^2} b(4n+1) =$$

$$\frac{1 \times 3}{2^2} \frac{3 \times 5}{4^2} \cdots \frac{(2n-1)(2n+1)}{(2n)^2} \frac{b(4n+1)}{2n+1}.$$

That is

$$b(1) = (\frac{2}{\pi} + o(1)) \frac{b(4n+1)}{2n+1}.$$

But $s+2 < b(s+2)$ and the functional equation yields:

$$s < b(s) < \frac{s^2+2s+1}{s+2} = s + \frac{1}{s+2}.$$

Hence $b(1) = \frac{2}{\pi}$.

For the continued fraction we start with the formal series:

$$b(s) = s + c_0 + \frac{c_1}{s} + \frac{c_2}{s^2} + \dots$$

and find the coefficients by applying Euclid division algorithm to series in $1/s$. Conclusion

$$b(s) = s + \frac{1^2}{2s + \frac{3^2}{2s + \frac{5^2}{2s + \dots}}}.$$

With convergence derived from the functional equations and elementary inequalities.

Closed form

$$b(s) = 4[\frac{\Gamma(3 + s/4)}{\Gamma(1 + s/4)}]^2.$$

due to Ramanujan.

The new "continuum"

Laurent series

$$f(z) = \sum_{k\in\mathbb{Z}} \frac{c_k}{z^k},$$

with finitely many $k < 0$ terms. Define

$$[f] = \sum_{k\leq 0} \frac{c_k}{z^k}, \quad \mathrm{Frac}(f) = f - [f].$$

And non-archimedean norm

$$\|f\| = \exp \deg f, \quad \deg(0) = -\infty.$$

The algorithm

Initial $f_0 = f$ produces a *P-fraction*:

$$f = [f_0] + \frac{1}{1/\mathrm{Frac}[f_0]} = [f_0] + \frac{1}{[f_1] + \frac{1}{[f_2]+\ldots}}$$

For non-rational $f(z)$ one finds $\deg Q_n \to \infty$ and

$$\|f - \frac{P_n}{Q_n}\| = \exp(-\deg Q_n - \deg Q_{n+1}).$$

Theorem. (Markov, Chebyshev, Gauss) An irreducible rational fraction P/Q is a convergent for the Laurent series f if and only if

$$\deg(f - P/Q) \leq -2\deg Q - 1.$$

Padé approximation

Specializes to Padé approximation problem: *given* f *and* $n > 0$ *find all polynomials* $P, Q,\ Q \neq 0, \deg Q \leq n$ *such that*

$$\deg(Qf - P) \leq n - 1.$$

A *normal index* is $\deg Q$ for an approximant P/Q.

Constructive aspects

Starting with $f(z) = [f](z) + \sum_{k\geq 1} \frac{c_k}{z^k}$ one defines:

$$H_n(f) = \begin{pmatrix} c_1 & c_2 & \dots c_n \\ c_2 & c_3 & \dots c_{n+1} \\ \vdots & \ddots & \\ c_n & c_{n+1} & \dots c_{2n-1} \end{pmatrix},$$

and

$$J_n(z) = \det \begin{pmatrix} c_1 & c_2 & \dots c_{n+1} & \\ c_2 & c_3 & \dots c_{n+2} & \\ \vdots & & \ddots & \\ c_n & c_{n+1} & \dots c_{2n} & \\ 1 & z & \dots & z^n \end{pmatrix}.$$

Main Theorem.

C. G. J. Jacobi (approx. 1850): An integer $n > 0$ is a normal index for f if and only of $\det H_n(f) \neq 0$. In that case the convergent P/Q with $\deg Q = n$ is, with a constant γ:

$$Q_n(z) = \gamma J_n(z)$$

$$P_n(z) = [J_n(z) f(z)].$$

And $f(z)$ is rational if and only if there exists N with $\det H_n(f) = 0, \quad n \geq N$. (Kronecker)

Abelian integrals

Let $R \in \mathbb{C}[z]$ of degree $\deg R = 2g + 2 \geq 2$ without multiple roots. Pell's type equation:

$$P^2 - Q^2 R = 1$$

has polynomial solutions, $Q \neq 0$ if and only if $\sqrt{R(z)}$ admits a periodic polynomial continued fraction expansion, if and only if there exists $r \in \mathbb{C}[z], \deg r = g$, so that

$$\int \frac{r}{\sqrt{R}} dz$$

can be expressed in elementary functions. (Abel 1826).

References

Perron, Oskar. *Die Lehre von den Kettenbrüchen*, Band I, Band II, Teubner, Stuttgart, 1954, 1957.

Khrushchev, Sergey. *Orthogonal polynomials and continued fractions. From Euler's point of view.* Encyclopedia of Mathematics and its Applications, 122. Cambridge University Press, Cambridge, 2008.

Stieltjes

Stieltjes has accumulated examples and computations concerning semi-convergent series ("the curse of divergent series" according to Abel), leading to a rigorous study of functions of s of the form:

$$b_0 s + c_0 + \mathbb{K}_{k=1}^{\infty}\left(\frac{a_n}{b_n s + c_n}\right),$$

where

$$a_n > 0, b_n \geq 0, \Re c_n \geq 0.$$

Main observation, for $s > 0$:

$$w \mapsto \frac{a_n}{b_n s + c_n + w}$$

preserves $\Re w > 0$.

Complex Markov Convergence Test

If

$$b_0 s + c_0 + \mathbb{K}_{k=1}^{\infty}\left(\frac{a_n}{b_n s + c_n}\right),$$

converges to finite values on a subset of $(0, \infty)$ with an accumulation point, then it converges to an analytic function defined on $\Re w > 0$.

Major advance, a la normal family argument, discovered by Stieltjes many years before Vitali. See his letters to Hermite.

Stieltjes Memoir-1894

$$S(z) = \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cfrac{1}{a_4 + \dots}}}},$$

with all parameters $a_j \geq 0$. Produces convergents satisfying

$$\lim_n \frac{P_{2n}(z)}{Q_{2n}(z)} = F(z),$$

$$\lim_n \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = F_1(z),$$

where F, F_1 are analytic functions on $\mathbb{C} \setminus (-\infty, 0]$.

Asymptotic expansion

$$S(z) \sim \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \ldots.$$

That is

$$\lim_{s\to\infty} s^{n+1}[S(s) - \frac{c_0}{s} + \frac{c_1}{s^2} + \ldots + (-1)^n \frac{c_{n-1}}{s^n}] = (-1)^n c_n,$$

for all $n \geq 1$.

Indeterminateness

Assume $\sum_n a_n < \infty$. Then all limits

$$\lim P_{2n}(z) = p(z), \quad \lim Q_{2n}(z) = q(z),$$

$$\lim P_{2n+1}(z) = p_1(z), \quad \lim Q_{2n+1}(z) = q_1(z),$$

exist in $\mathbb{C}$ and

$$\frac{p(z)}{q(z)} = \sum_j \frac{M_j}{z + m_j}$$

with $M_j > 0, m_j \geq 0$. Similarly for $\frac{p_1(z)}{q_1(z)}$, but

$$\frac{p(z)}{q(z)} \neq \frac{p_1(z)}{q_1(z)}.$$

Determinateness

Assume $\sum_n a_n = \infty$. Then the continued fraction converges on $\mathbb{C} \setminus (-\infty, 0]$ to a function of the form

$$S(z) = \int_0^\infty \frac{df(u)}{u+z},$$

where f is monotonically non-decreasing on $[0, \infty)$ and all power moments of $df(u)$ exist. A new notion of integral was developed by Stieltjes for this purpose. Unifying in particular the two cases

$$\sum_j \frac{M_j}{z+m_j} = \int_0^\infty \frac{d\phi(u)}{u+z}.$$

Comparison: old continuum versus the new one

$$\xi = \sum_{j=0}^{\infty} \frac{n_j}{10^j} \in [0, \infty), \quad \xi = b_0 + \mathbb{K}_j(\frac{1}{b_j}),$$

and

$$\sigma \in \mathrm{Meas}_+[0, \infty), \quad \int_0^\infty \frac{d\sigma(u)}{u+z} \sim \frac{1}{a_1 z + \frac{1}{a_2 + \frac{1}{a_3 z + \frac{1}{a_4 + \ldots}}}},$$

In both representations the parameters $b_j \in \mathbb{Z}_+$, respectively $a_j \geq 0$ are *independent*.

Cauchy transforms

There is a *constructive bijective* correspondence between analytic functions $F(z)$ defined in the half-plane $\Im(z) > 0$ and satisfying $\Im F(z) > 0$ there, and positive Borel measures σ of finite mass, defined on $\mathbb{R}$ admitting all power moments:

$$F(z) = az + b + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\sigma(t),$$

where $a \geq 0$ and $b \in \mathbb{R}$.
Moreover, the measure σ admits all moments, if and only if

$$\sup_{y \geq 1} |yF(iy)| < \infty$$

and there are real numbers $s_k, \ k \geq 0$, satisfying

$$\lim_{y \to \infty} (iy)^{2n+1} [F(iy) + \frac{s_0}{iy} + \frac{s_1}{(iy)^2} + \ldots + \frac{s_{2n-1}}{(iy)^{2n}}] = -s_{2n}.$$

Recovery: Stieltjes and Perron

If

$$F(z) = \int_{-\infty}^{\infty} \frac{1}{t-z} d\sigma(t), \ \Im z > 0,$$

then

$$\frac{\sigma\{a\} + \sigma\{b\}}{2} + \sigma(a, b) = \lim_{\epsilon \to 0} \frac{1}{\pi} \Im \int_a^b F(x + i\epsilon) dx.$$

Hamburger articles 1919-1921

Real heritor of Stieltjes, author of a detailed study of the moment problem on the real line. All centered on continued fraction expansions of the form:

$$S(z) = \frac{a_1}{z + b_1 + \frac{a_2}{z+b_2+\frac{a_3}{z+b_3+\dots}}},$$

with $a_j \neq 0$ real and b_j complex.
Convergence assured by selection principle of Grommer (à la normal families argument).
Parametrization of all solutions of the truncated problem.
Identification, and recognition of importance, of *Christoffel and Darboux kernel*.
Complemented by R. Nevanlinna function theoretic study (1922) of Stieltjes moment problem.

Polya 1920 and limit theorems of probability theory

Theorem.(Continuity of the Moment Problem) Assume $f(x)$ is a continuous, non-decreasing distribution function on the real line, admitting all moments;

$$s_n = \int_{-\infty}^{\infty} t^n df(t), \quad n \geq 0,$$

with $\limsup \frac{s_{2n}^{1/(2n)}}{n} < \infty$. If a sequence of distribution functions $f_k(x)$ satisfies

$$\lim_k \int_{-\infty}^{\infty} t^n df_k(t) = s_n, \; n \geq 0,$$

then

$$\lim_k f_k(x) = f(x)$$

uniformly on compact subsets of the real line.

Century long quest, paved with frustrations and failures, to prove the conjectured Central Limit Theorem stated by Laplace.

One of the main motivations of Chebyshev and A. Markov to study the moment problem. And later Lyapunov. They did it initially for the Gaussian distribution as limit.

Tutorial: Proofs of Seidel and Koch tests

Assume $b_n > 0$ for all n. Then (Seidel)

$$\mathbb{K}_1^\infty(\frac{1}{b_n}) \text{ converges} \Leftrightarrow \sum_1^\infty b_n = \infty.$$

Recall that the convergents $\frac{P_n}{Q_n}$ satisfy:

$$P_n = b_n P_{n-1} + P_{n-2}, \quad Q_n = b_n Q_{n-1} + Q_{n-2},$$

$$P_{-1} = 1,\ P_0 = 0,\ Q_{-1} = 0,\ Q_0 = 1,$$

and

$$\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} = \frac{1}{Q_{2n} Q_{2n+1}}.$$

1) Prove:

$$\frac{Q_n}{Q_{n-1}} = b_n + \frac{1}{b_{n-1} + \frac{1}{b_{n-2}+\ldots+\frac{1}{b_1}}}.$$

2) Show that

$$\frac{Q_n}{Q_{n-2}} = 1 + b_n[b_{n-1} + \frac{1}{b_{n-2} + \ldots + \frac{1}{b_1}}] > 1.$$

3) Deduce $Q_{2n} > Q_{2n-2} > \ldots > 1, \;\; Q_{2n+1} > Q_{2n-1} > \ldots > b_1$. Hence

$$Q_{2n} > b_1(b_2 + b_4 + \ldots + b_{2n}),$$

and

$$Q_{2n+1} > b_{2n+1} + b_{2n-1} + \ldots + b_1.$$

4) If $\sum b_n = \infty$, then $\mathbb{K}_1^\infty(\frac{1}{b_n})$ converges.

If $\sum b_n < \infty$, use Koch criterion (next).

Proof of Koch criterion

Assume $b_n \in \mathbb{C}$ satisfy $\sum_n |b_n| < \infty$. Then

$$\lim_n P_{2n} = P, \quad \lim_n P_{2n+1} = P',$$
$$\lim_n Q_{2n} = Q, \quad \lim_n Q_{2n+1} = Q',$$

and

$$P'Q - PQ' = 1.$$

1) Can assume $P_0 = 1$, so that (by induction):

$$|P_n| \leq (1 + |b_1|)(1 + |b_2|) \ldots (1 + |b_n|), \quad n \geq 1.$$

and similarly $|Q_n|$.
Hence the sequences $(|P_n|)$ and $(|Q_n|)$ are bounded.

2) Prove

$$P_{2n} = b_{2n} P_{2n-1} + b_{2n-2} P_{2n-3} + \ldots + b_2 P_1 + P_0,$$

and deduce that $\lim_n P_{2n}$ exists. Similarly for the other three sub-sequences.

3) Infer

$$P' Q - P Q' = 1.$$