Polynomial optimization (revisited): references and exercises

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1 References

The moment-SOS hierarchy applied to polynomial optimization is described in [12, 13] and surveyed in [11], see also [14] for an introduction. It fits the framework of the generalized problem of moments [15, 10] that we also follow in the second and third lecture to deal with dynamical systems.

See [2, 1, 4] for textbooks on convex optimization. Conic duality and infinite-dimensional optimization are covered in [16] and also [1].

Asymptotic convergence of the moment-SOS hierarchy for POP, as proved originally in [13], relies on Putinar's solution [19] to the problem of moments based on a version of the Positivstellensatz, a representation of positive polynomials [20]. See also [17, 3] for positive polynomials and SOS.

Finite convergence, and global optimality certificate for POP is based on flat extensions of moment matrices [5, 6]. Generic finite convergence of the moment-SOS hierarchy was proved in [18].

Extraction of the global minimizers for POP was described in [8].

The GloptiPoly package for Matlab is described in [7, 9].

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2 Exercises

2.1 Exercise 1.1

2.1.1 Statement

Given a polynomial $p \in \mathbb{R}[x]$ and a compact set X, the polynomial optimization problem (POP)

$$\begin{array}{rcll} v^* & = & \min_x & p(x) \\ & & \text{s.t.} & x \in X \end{array}$$

is reformulated as the linear problem (LP)

$$\begin{array}{rcll} p^* & = & \inf_\mu & \langle p, \mu \rangle \\ & & \text{s.t.} & \langle 1, \mu \rangle = 1 \\ & & & \mu \in C(X)'_+ \end{array}$$

where $C(X)'_+$ is the cone of (Borel regular positive) measures on X, topologically dual to the cone of positive functions on X, and the duality

$$\langle f, \mu \rangle := \int_X f(x) d\mu(x)$$

is integration of a function f by a measure μ.

Prove that $v^* = p^*$ and that the LP has an optimal solution equal to the Dirac measure at any optimal solution of the POP.

2.1.2 Solution

To be completed.

2.2 Exercise 1.2

2.2.1 Statement

Dual to the primal LP

$$\begin{array}{rcll} p^* & = & \inf_\mu & \langle p, \mu \rangle \\ & & \text{s.t.} & \langle 1, \mu \rangle = 1 \\ & & & \mu \in C(X)'_+ \end{array}$$

is the LP

$$\begin{array}{rcll} d^* & = & \sup_{v \in \mathbb{R}} & v \\ & & \text{s.t.} & p - v \in C(X)_+. \end{array}$$

Derive the dual LP from the primal LP using convex duality. Prove that strong duality holds i.e. $p^* = d^*$. Give a graphical interpretation to the dual LP.

2.2.2 Solution

To be completed.

2.3 Exercise 1.3

2.3.1 Statement

Prove that deciding whether a polynomial is a sum of squares (SOS) can be reduced to semidefinite programming.

2.3.2 Solution

To be completed.

2.4 Exercise 1.4

Given polynomials $q_k \in \mathbb{R}[x]$, $k = 0, 1, \ldots, m$ and an integer d, define for integer $r \geq d$ the truncated quadratic module

$$Q(X)_{d,r} := \{p \in \mathbb{R}[x]_d : p = \sum_{k=0}^{m} s_k q_k, \;\; s_k \in \Sigma, \;\; s_k q_k \in \mathbb{R}[x]_r\}$$

as a projection of the SOS cone Σ, where $\mathbb{R}[x]_r$ denotes the vector space of polynomials of degree up to r.

Describe explicitly the moment cone relaxation $Q(X)'_{d,r}$ as the projection of a spectrahedron.

2.4.1 Solution

To be completed.

2.5 Exercise 1.5

Consider the primal conic problem

$$\begin{array}{rcll} p_r^* & = & \inf_y & \langle p, y \rangle \\ & & \text{s.t.} & \langle 1, y \rangle = 1 \\ & & & y \in Q(X)'_{d,r} \end{array}$$

and its dual conic problem

$$\begin{array}{rcll} d_r^* & = & \sup_{v \in \mathbb{R}} & v \\ & & \text{s.t.} & p - v \in Q(X)_{d,r}. \end{array}$$

Prove that strong duality holds: $p_r^* = d_r^*$.

2.5.1 Solution

To be completed.

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