Positively invariant set estimation: references, exercises, questions and answers

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Online learning weeks of the POEMA network

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1 References

The lecture follows closely [3], where all the proofs are detailed, in the case of controlled dynamical systems, both in continuous-time and discrete-time. Many of the technical arguments are adaptations of ideas proposed previously in [2] for approximating the region of attraction of controlled ordinary differential equations. For further developments see [6].

The technical background on push-forward measures, Koopman and Frobenius-Perron operators for dynamical systems is covered in [4]. In particular, the example of push-forward measure for the logistic map is described in [4, Section 1.2].

The moment-SOS hierarchy was applied in [1] for computing (moments of) invariant measures (fixed points of the Frobenius-Perron operator) of one-dimensional dynamical systems, in particular for the logistic map. See [5] for computing invariant measures with the hierarchy from a broader perspective

2 Exercises

2.1 Exercise 2.1

2.1.1 Statement

Consider the logistic map

$$f(x) = 4x(1-x)$$

on

$$X := [0, 1].$$

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a. Given $\mu(dx) = m_0(x)dx$, derive analytically $f_{\#}\mu$.

b. Given $\mu(dx) = I_{[0,1]}(x)dx$ compute $f_{\#}\mu$ and $f \circ f_{\#}\mu$.

c. Prove that $\mu(dx) = dx/(\pi\sqrt{x(1-x)})$ is invariant.

d. Prove that $\mu(dx) = \delta_{3/4}(dx)$ is invariant.

2.1.2 Solution

To be completed.

2.2 Exercise 2.2

2.2.1 Statement

Consider the LP

$$\begin{array}{rcll} p^* & = & \sup & \langle 1, \mu_0 \rangle \\ & & \text{s.t.} & \mu = \mu_0 + \alpha f_{\#}\mu \\ & & & \mu_0 + \hat{\mu}_0 = \lambda_X \end{array}$$

where λ_X is the Lebesgue measure on X and the optimization variables are μ, μ_0, $\hat{\mu}_0$ all in $C(X)'_+$.

Prove that the supremum is attained by $\mu_0^* = \lambda_{X_I}$ and hence $p^* = \text{vol}\, X_I$.

2.2.2 Solution

To be completed.

2.3 Exercise 2.3

2.3.1 Statement

Derive the dual LP

$$\begin{array}{rcll} d^* & = & \inf & \langle w, \lambda_X \rangle \\ & & \text{s.t.} & (v - \alpha v \circ f,\ w - v - 1,\ w) \in C(X)^3_+. \end{array}$$

by convex duality. Prove that there is no duality gap.

2.3.2 Solution

To be completed.

2.4 Exercise 2.4

2.4.1 Statement

By replacing $C(X)_+$ with $Q(X)_{r,r}$ we get a monotone converging sequence of lower bounds

$$p_r^* = d_r^* \leq p_{r+1}^* = d_{r+1}^* \leq p_\infty^* = d_\infty^* = \mathrm{vol}\, X_I.$$

Prove it with the Stone-Weierstrass Theorem.

2.4.2 Solution

To be completed.

2.5 Exercise 2.5

2.5.1 Statement

In the dual we obtain a sequence of polynomials v_r, w_r in $\mathbb{R}[x]_r$ such that

$$X_{Ir} := \{x \in X : v_r(x) \geq 0\} \supset X_I$$

and

$$\lim_{r\to\infty} \mathrm{vol}(X_{Ir} \setminus X_I) = 0.$$

Prove it by showing that $w_r \to I_{X_I}$ in $L_1(X)$.

2.5.2 Solution

To be completed.

3 Questions and answers

To be completed.

References

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[5] V. Magron, M. Forets, D. Henrion. Semidefinite approximation of invariant measures for polynomial systems. Discrete and Continuous Dynamical Systems Series B, 24(12):6745-6770, 2019.

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