Introduction to Christoffel-Darboux kernels for polynomial optimization

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Outline

1. CD kernel captures measure theoretic properties: univariate case

2. Quantitative asymptotics

3. The singular case

Pure point part of a measure

Exercise: Let μ be a compactly supported Borel probability measure on $\mathbb{R}^p$ and define Λ_d^μ, with its variational form. Show that

$$\lim_{d \to \infty} \Lambda_d^\mu(x_0) = \mu(\{x_0\}),$$

for all x_0 in $\mathbb{R}^p$.

Typical value

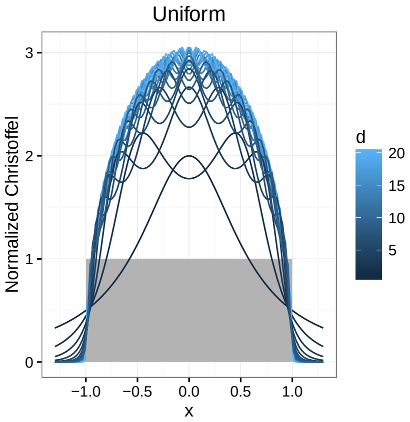
Exercise: Let μ be a compactly supported absolutely continuous probability measure on $\mathbb{R}^p$. Let Z be a random variable with distribution μ, show that

$$\mathbb{E}_{Z\sim\mu}\left[(\Lambda_d^\mu(Z))^{-1}\right] = \binom{d+p}{p} \sim d^p.$$

Asymptotics for the Christoffel function: sublinear on the support

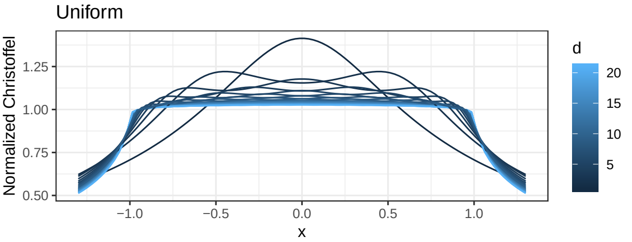
Maté, Nevai and Totik, (1991): $p = 1$ and $d\mu = f$ on $[-1, 1]$ and 0 elsewhere, $f > 0$ continuous. For almost all x in $[-1, 1]$

$$\lim_{d \to \infty} \Lambda_d^{\mu}(x) d = \pi f(x) \sqrt{1 - x^2}$$



Stahl and Totik, (1992): $p = 1$ and $d\mu = f$ on $[-1, 1]$ and 0 elsewhere, $f > 0$, for all $x \notin [-1, 1]$,

$$\lim_{d \to \infty} \Lambda_d^{\mu}(x)^{\frac{1}{2d}} < 1$$



In a nutshel

Exponential growth dichotomy: Growth of the CD kernel is

- At most polynomial in the degree d in the interior of the support.
- Exponential in the degree d outside the support.

Asymptotics after rescaling: involves the product of a density term and a term which is specific to the support.

Generalizations:

- In dimension 1 to more complex measures (Totik).
- In higher dimensions and non euclidean settings were recently described in connection with pluripotential theory (Bloom, Berman, Boucksom, Nystrom, Shiffman)

Outline

Explicit construction: the cube

Legendre Polynomials: $P_0(x) = 0$, $P_1(x) = x$

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$\max_{x\in[-1,1]} P_n(x) = 1$.

Orthogonality:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}.$$

Lebesgue measure on the cube: orthogonal polynomials given by

$$Q_\alpha(\mathbf{x}) = \prod_{i=1}^{p} \sqrt{\alpha_i + \frac{1}{2}} P_{\alpha_i}(x_i), \qquad \alpha \in \mathbb{N}_+^p, \quad |\alpha| < d$$

Let λ_C be the restriction of Lebesgue measure to the unit cube $C = [-1,1]^p$, then

$$\sup_{\mathbf{x}\in C} K_d^{\lambda_C}(\mathbf{x},\mathbf{x}) \leq \sum_{|\alpha|\leq d} \prod_{i=1}^{p} \left(\alpha_i + \frac{1}{2}\right) = O(d^{2p})$$

The unit euclidean ball (Bos, Xu)

ω_p is the area of the p dimensional unit sphere in $\mathbb{R}^{p+1}$.

Lebesgue measure on the ball: Let λ_B be the restriction of Lebesgue measure to the unit Euclidean ball $B \subset \mathbb{R}^p$. We have

$$K_d^{\lambda_B}(0,0) \leq \frac{s(d)}{\omega_p} \frac{(d+p+1)(d+p+2)(2d+p+6)}{(d+1)(d+2)(d+3)} = O(d^p)$$

$$K_d^{\lambda_B}(\mathbf{x},\mathbf{x}) = 2\binom{p+d+1}{d} - \binom{p+d}{d} = O(d^{p+1}), \qquad \|\mathbf{x}\| = 1.$$

Let w be a positive density on the unit ball $B \subset \mathbb{R}^p$, Lipschitz and symmetric, and μ the corresponding measure then locally uniformly on compact subsets in $\mathrm{int}(B)$

$$\lim_{d\to\infty} s(d)\Lambda_d^{\mu}(x) = w(x)\frac{\omega_p}{2}\sqrt{1-\|x\|^2}.$$

Exercise: Show that if $\mu(A) \geq \nu(A)$ for all measurable set A, then for all d, $K_d^{\mu} \leq K_d^{\nu}$.

Lebesgue measure on a set with non empty interior: Let $S \subset \mathbb{R}^p$ have non empty interior. Then for all $\mathbf{x} \in \mathrm{int}(S)$,

$$K_d^{\lambda_S}(\mathbf{x}, \mathbf{x}) = O(d^p)$$

If in addition the boundary of $S \subset \mathbb{R}^p$ is a smooth embedded hypersurface in $\mathbb{R}^p$. Then

$$\sup_{\mathbf{x} \in S} K_d^{\lambda_S}(\mathbf{x}, \mathbf{x}) = O(d^{p+1}).$$

Tubular neighborhood theorem: There exists $r > 0$ such that for all $\mathbf{x} \in S$, there is a ball of radius r, $B_r \subset S$ such that $x \subset B_r$.

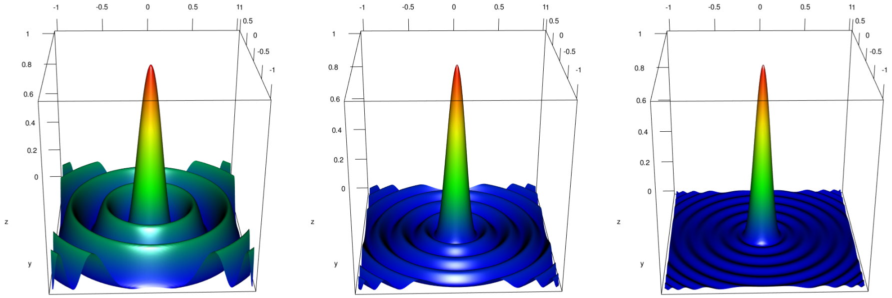
Let $S \subset \mathbb{R}^p$ be compact and μ be a probability measure supported on S. Then for all $\mathbf{x}$ with $\mathrm{dist}(\mathbf{x}, S) \geq \delta > 0$, and $d \in \mathbb{N}$

$$K_d^{\mu}(\mathbf{x}, \mathbf{x}) \geq 2^{\frac{\delta d}{\delta + \mathrm{diam}(S)} - 3}.$$

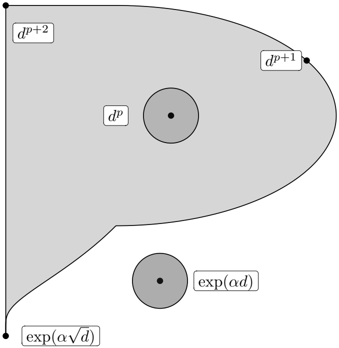
Kroó's needle polynomial, for any $\delta > 0$, $d \in \mathbb{N}^*$, $\exists Q \in \mathbb{R}_{2d}[X]$

$$Q(0) = 1, \qquad |Q(\mathbf{x})| \leq 1 \text{ if } \|\mathbf{x}\| \leq 1, \qquad |Q(\mathbf{x})| \leq 2^{1-\delta d} \text{ if } \delta \leq \|\mathbf{x}\| \leq 1.$$

Example for $\delta = 0.2$ and $d = 20, 30, 40$.



μ: Lebesgue restricted to $S \subset \mathbb{R}^p$, compact, non-empty interior.



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3. The singular case

μ: Borel probability measure in $\mathbb{R}^p$, compact support S, absolutely continuous.
$\mathbb{R}_d[X]$: p-variate polynomials of degree at most d (of dimension $s(d) = \binom{p+d}{d}$).

$$(P, Q) \quad \mapsto \quad \langle\langle P, Q \rangle\rangle_\mu := \int PQd\mu,$$

defines a valid scalar product on $\mathbb{R}_d[X]$.a positive semidefinite bilinear form on $\mathbb{R}_d[X]$.

Specificity of the singular case

μ: Borel probability measure in $\mathbb{R}^p$, asbolutely continuous, compact support: S.
$\mathbb{R}_d[X]$: p-variate polynomials of degree at most d (of dimension $s(d) = \binom{p+d}{d}$).

Moment based computation

- Let $\{P_i\}_{i=1}^{s(d)}$ be any basis of $\mathbb{R}_d[X]$,
- $\mathbf{v}_d : \mathbf{x} \mapsto (P_1(\mathbf{x}), \ldots, P_{s(d)}(\mathbf{x}))^T$.
- $M_{\mu,d} = \int \mathbf{v}_d \mathbf{v}_d^T d\mu \in \mathbb{R}^{s(d)\times s(d)}$.

Then, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, $K_d^\mu(\mathbf{x}, \mathbf{y}) = \mathbf{v}_d(\mathbf{x})^T M_{\mu,d}^{-1} \mathbf{v}_d(\mathbf{y})$ ~~$\mathbf{v}_d(\mathbf{x})^T M_{\mu,d}^{-1} \mathbf{v}_d(\mathbf{y})$~~

Let $P \in \mathbb{R}_d[X]$ such that $P(\mathbf{x}) = \sum_{i=1}^{s(d)} \mathbf{p}_i P_i(\mathbf{x})$. We have

$$\int P^2 d\mu = \mathbf{p}^T M_{\mu,d} \mathbf{p}.$$

If P vanishes on S, then $\mathbf{p} \in \ker(M_{\mu,d})$,
The moment matrix is singular.
Morally, the CD kernel should be infinite

Christoffel function to the rescue

μ: Borel probability measure in $\mathbb{R}^p$, asbolutely continuous, compact support: S.
$\mathbb{R}_d[X]$: p-variate polynomials of degree at most d (of dimension $s(d) = \binom{p+d}{d}$).

Variational formulation: for all $\mathbf{z} \in \mathbb{R}^p$

$$\frac{1}{K_d^\mu(\mathbf{z},\mathbf{z})} = \Lambda_d^\mu(\mathbf{z}) = \min_{P \in \mathbb{R}_d[X]} \left\{ \int P^2 d\mu : \quad P(\mathbf{z}) = 1 \right\}.$$

$$\Lambda_d^\mu(\mathbf{z}) = \min_{P \in \mathbb{R}_d[X]} \left\{ \int P^2 d\mu : \quad P(\mathbf{z}) = 1 \right\}.$$

Given $\mathbf{z} \in \mathbb{R}^p$, such that there exists $P \in \mathbb{R}_d[X]$ such that

- $P(\mathbf{z}) \neq 0$
- P vanishes on S.

Then $\Lambda_d^\mu(\mathbf{z}) = 0$.

Getting the CD kernel back (and computation from moments)

μ: Borel probability measure in $\mathbb{R}^p$,compact support: S.
$\mathbb{R}_d[X]$: p-variate polynomials of degree at most d (of dimension $s(d) = \binom{p+d}{d}$).
V denotes the Zariski closure of S (smallest algebraic set containing S).

For d large enough, $V = \{\mathbf{z} \in \mathbb{R}^p, \Lambda_d^\mu(\mathbf{z}) > 0\}$.

Polynomials on V**:** $L^2_{\mu,d} = \mathbb{R}_d[X] \ / \ \{P \in \mathbb{R}_d[X],\, P \text{ vanishes on } V\}$.

RKHS: $(L^2_{\mu,d}, \langle\langle \cdot, \cdot \rangle\rangle_\mu)$ is a Hilbert space of functions on V. K_d^μ is its reproducing kernel (defined on V).
For any $\mathbf{x} \in V$ and $P \in L^2_{\mu,d}$, $P(\mathbf{x}) = \int P(\mathbf{y}) K_d^\mu(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y})$.

Relation with Christoffel function: $\Lambda_d^\mu(\mathbf{z}) K_d^\mu(\mathbf{z}, \mathbf{z}) = 1$, for $\mathbf{z} \in V$.

Pseudo inverse computation: let $\mathbf{v}_d$ be any basis of $\mathbb{R}_d[X]$, $M_{\mu,d}$ moment matrix:

$$\forall \mathbf{x}, \mathbf{y} \in V \qquad K_d^\mu(\mathbf{x}, \mathbf{y}) = \mathbf{v}_d(\mathbf{x}) M_{\mu,d}^\dagger \mathbf{v}_d(\mathbf{y}).$$

$$\forall \mathbf{z} \in \mathbb{R}^p \qquad \Lambda_d^\mu(\mathbf{z}) = \begin{cases} 0 & \text{if } \mathrm{proj}_{\ker(M_{\mu,d})}(\mathbf{v}_d(\mathbf{z})) \neq 0 \\ \left(\mathbf{v}_d(\mathbf{z}) M_{\mu,d}^\dagger \mathbf{v}_d(\mathbf{z})\right)^{-1} & \text{otherwise.} \end{cases}$$