An introduction to Christoffel-Darboux kernels for polynomial optimization

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1 Source and references

The univariate asymptotics results are found in Maté, Nevai, Totik [5] and in the book of Stahl and Totik [7], which offers a much broarder view on the topic and illustrates the relations with potential theory. A more general result is found in [8]. The unit cube is discussed in [9] The multivariate results on the euclidean ball are found in works by Bos, Xu and co-authors [1, 2, 10, 11]. The needle Polynomial was introduced by Kroo-Lubinsky [3]. The course was inspired from content developped in [4, 6].

2 Exercises

Exercise 1 (Pseudo inverse formula)**.**

1. *Let M be a symmetric definite positive $p \times p$ matrix and $u \in \mathbb{R}^p$, $u \neq 0$*

$$\min_{x \in \mathbb{R}^p} \left\{ x^T M x, \text{ s.t. } \quad x^T u = 1 \right\} = \frac{1}{u^T M^{-1} u}.$$

and that the minimum is attained at $x_0 = \frac{M^{-1}u}{u^T M^{-1} u}$.

2. *Let M be a symmetric semidefinite positive $p \times p$ matrix and $u \in \mathbb{R}^p$, $u \notin \mathrm{Im}(M)$, show that*

$$\min_{x \in \mathbb{R}^p} \left\{ x^T M x, \text{ s.t. } \quad x^T u = 1 \right\} = 0.$$

3. *Let M be a symmetric semidefinite positive $p \times p$ matrix and $u \in \mathbb{R}^p$, $u \in \mathrm{Im}(M)$, show that*

$$\min_{x \in \mathbb{R}^p} \left\{ x^T M x, \text{ s.t. } \quad x^T u = 1 \right\} = \frac{1}{u^T M^{\dagger} u}.$$

where $M^\dagger$ denotes Moore-Penrose pseudo-inverse of M.

4. *Prove the pseudo inverse formula for the Christoffel-Darboux kernel in the singular case.*

Exercise 2 (Bound on the cube)**.**

1. *Show that for any integers* $1 \leq k \leq n$

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

2. *Let* N_d *be the number of monomials of degree exactly* d *in* p *variable, show that as* $d \to \infty$.

$$N_d = O(d^{p-1})$$

3. *Show the following inequality for sums over multi indices.*

$$\sum_{|\alpha| \leq d} \prod_{i=1}^{p} \left(\alpha_i + \frac{1}{2}\right) \leq \sum_{|\alpha| \leq d} \left(\frac{|\alpha| + \frac{p}{2}}{p}\right)^p.$$

4. *Deduce that as* $d \to \infty$

$$\sum_{|\alpha| \leq d} \prod_{i=1}^{p} \left(\alpha_i + \frac{1}{2}\right) = O(d^{2p}).$$

Exercise 3 (Needle polynomial)**.** *For all* $d \in \mathbb{N}$*,* T_d *denotes the univariate Chebyshev polynomial of the first kind. We recall that for* $t \geq 1$*,*

$$T_d(t) = \frac{1}{2}\left(\left(t + \sqrt{t^2 - 1}\right)^d + \left(t + \sqrt{t^2 - 1}\right)^{-d}\right).$$

Furthermore, $T_d(t) \in [-1, 1]$ *for* $t \in [-1, 1]$*,* $T_d(1) = 1$ *and* T_d *is increasing on* $(1, +\infty)$*.*

1. *Sow that for all* $\delta > 0$*,*

$$1 + \delta^2 + \sqrt{(1 + \delta^2)^2 - 1} \geq 1 + \sqrt{2}\delta$$

2. *Using concavity of the logarithm, deduce that for all* $0 \leq \delta \leq 1$*,*

$$T_d(1 + \delta^2) \geq 2^{\delta d - 1},$$

3. *show that for all* $d \in \mathbb{N}$*,*

$$Q \colon x \mapsto \frac{T_d(1 + \delta^2 - \|x\|^2)}{T_d(1 + \delta^2)}$$

satisfies the properties of the needle polynomial

Exercise 4. *Consider the set* $S = \left\{(x, y) \in \mathbb{R}^2, 0 \leq x \leq 1, 0 \leq y \leq e^{-1/x}, x^2 + y^2 \leq 1\right\}$ *with the convention that* $(0, 0) \in S$ *and* $d\mu$ *the Lebesgue measure restricted to* S*. Show that for any* $d \in \mathbb{N}^*$*,*

$$K_d^\mu(0, 0) \geq \frac{e^{\sqrt{d}}}{5}.$$

References

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