An introduction to Christoffel-Darboux kernels for polynomial optimization

Edouard Pauwels

June 17, 2020

1 Source and references

The Christoffel-Darboux kernel is named after the work of Jean Gaston Darboux [2] and Elwin Bruno Christoffel [1]. A classical reference on orthogonal polynomials is Szegö's book [8] for the univariate case. A detailled treatment of the multivariate setting has come much later, an important reference being the book of Dunkl and Xu [3]. Many more information and bibliographical comment regarding the Christoffel-Darboux kernel can be found in the overview provided by Simon [6], while an account of the important role played by the Christoffel function in modern analysis and approximation theory is given in [5]. The asymptotics results are found in Maté, Nevai, Totik [4] and in the book of Stahl and Totik [7], which offers a much broarder view on the topic and illustrates the relations with potential theory.

2 Exercises

Exercise 1 (A valid scalar product). *A measure μ on $\mathbb{R}^p$ is called polynomial determining if for any $d \in \mathbb{N}$ and any $P \in \mathbb{R}_d[X]$, $P \neq 0$, $\mu\{x \in \mathbb{R}^p,\ P(x) = 0\} = 0$.*

1. *Let the support of μ be the restriction of Lebesgue measure to a set with non-empty interior, show that μ is polynomial determining.*
2. *Let μ be absolutely continuous with respect to Lebesgue measure, prove that μ is polynomial determining.*
3. *Formulate minimal conditions on μ so that $\langle\langle \cdot, \cdot \rangle\rangle_\mu$ defines a valid scalar product on $\mathbb{R}_d[X]$ for any $d \in \mathbb{N}$.*

Exercise 2. *Let μ be a compactly supported absolutely continuous probability measure on $\mathbb{R}^p$. Let Z be a random variable with distribution μ, show that*

$$\mathbb{E}_{Z\sim\mu}\left[(\Lambda_d^\mu(Z))^{-1}\right] = \binom{d+p}{p} \sim d^p.$$

Exercise 3 (Christoffel-Darboux formula and orthogonal polynomials in one variable). *Consider the univariate case, μ a compactly supported measure with a nonzero absolutely continuous part (assume that μ has a density with respect to Lebesgue's measure for simplicity), and $(P_i)_{i\in\mathbb{N}}$ a sequence of orthonormal polynomials (with respect to $\langle\langle \cdot, \cdot \rangle\rangle_\mu$), P_i of degree i, for all $i \in \mathbb{N}$.*

1. *Show that for all $n \in \mathbb{N}$ and all $i \in \mathbb{N}$ with $i < n$ and Q of degree i, $\int P_n(x)Q(x)d\mu(x) = 0$.*

2. *For all $i \in \mathbb{N}$, set k_i the leading coefficient of P_i, show that for all $i \in \mathbb{N}$,*

$$\int x P_i(x) P_{i+1}(x) = \frac{k_i}{k_{i+1}}.$$

3. *For all $i \in \mathbb{N}$, set $a_i = k_i/k_{i+1}$ and $b_i = \int x P_i(x)^2 d\mu(x)$. Show That for all $n > 0$, and all $x \in \mathbb{R}$,*

$$x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x).$$

4. *Show that for all $x, y \in \mathbb{R}$, $x \neq y$, and all $n \in \mathbb{N}$,*

$$\sum_{i=0}^{n} P_i(x) P_i(y) = \frac{k_n}{k_{n+1}} \frac{P_{n+1}(x) P_n(y) - P_{n+1}(y) P_n(x)}{x - y},$$

this is the Christoffel-Darboux formula.

5. *Deduce that for all $x \in \mathbb{R}$,*

$$\frac{1}{\Lambda_n^\mu(x)} = \frac{k_n}{k_{n+1}} \left(P'_{n+1}(x) P_n(x) - P_{n+1}(x) P'_n(x) \right)$$

6. *Show that each P_i has i distinct real roots interlaced by those of P_{i+1} (exactly one zero of P_i between two consecutive zeros of P_{i+1}).*

Exercise 4 (Recovering the pure point part of a measure). *Let μ be a compactly supported probability measure on $\mathbb{R}^p$ and define Λ_d^μ, with its variational form. Show that*

$$\lim_{d \to \infty} \Lambda_d^\mu(x_0) = \mu(\{x_0\}),$$

for all x_0 in $\mathbb{R}^p$. Use the variational formulation, start with the univariate case.

Exercise 5 (Translation invariance). *In this exercise $\mathcal{A}$ is an invertible affine map on $\mathbb{R}^p$.*

1. *If $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$, where $x_i \in \mathbb{R}^p$, $i = 1, \ldots n$. Show that $\mathcal{A}_* \mu = \frac{1}{n} \sum_{i=1}^n \delta_{\mathcal{A}(x_i)}$.*
2. *If μ is a probability measure uniform on a bounded open set $\mathcal{O} \subset \mathbb{R}^p$ (proportional to Lebesgue measure), then $\mathcal{A}_* \mu$ is the restriction of Lebesgue measure to $\mathcal{A}(O)$.*
3. *Construct a measure on $\mathbb{R}^2$, "uniform" on a certain set S and an invertible affine map $\mathcal{A}$ such that $\mathcal{A}_* \mu$ is not "uniform" on $\mathcal{A}(S)$.*

References

[1] E. B. Christoffel (1858). Über die Gaußische Quadratur und eine Verallgemeinerung derselben. Journal für die reine und angewandte Mathematik, 55, 61-82.

[2] J. G. Darboux (1878). Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en sÃ©rie. Journal de Mathématiques pures et appliquées, 5-56.

[3] C. Dunkl and Y. Xu (2001). *Orthogonal polynomials of several variables.* Cambridge University Press.

[4] A. Máté, P. Nevai, and V. Totik (1991). Szego's extremum problem on the unit circle. *Annals of Mathematics*, 134(2):433–53.

[5] P. Nevai (1986). Géza Freud, orthogonal polynomials and Christoffel functions. A case study. *Journal of Approximation Theory*, 48(1):3–167.

[6] B. Simon (2008). The Christoffel-Darboux kernel. Proceedings of Symposia in Pure Mathematics (79), 295–335.

[7] H. Stahl, J. Steel and V. Totik (1992). General orthogonal polynomials (No. 43). Cambridge University Press.

[8] G. Szegö (1974). *Orthogonal polynomials.* In Colloquium publications, AMS, (23), fourth edition.